Homework 7

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Sunday 3rd November, 2024

1 About inverse

1 Given invertible matrix $M \in \mathbb{F}^{n\times n}$, if we rotate M by 90 degrees clockwise, what will happen to M^{-1}?

Solution.

M^{-1} will be rotated by 90 degrees anticlockwise.

2 Find the inverse of

$$M = \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 \\ -1 & 2 & -1 & \cdots & -1 \\ -1 & -1 & 2 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 2 \end{bmatrix}.$$

Solution.

Do row exchange to $\begin{bmatrix} M & I \end{bmatrix}$.

Subtract the first row from others to get

$$\begin{bmatrix} 1 & -1 & -1 & \cdots & -1 & 1 & 0 & 0 & \cdots & 0 \\ -2 & 3 & 0 & \cdots & 0 & -1 & 1 & 0 & \cdots & 0 \\ -2 & 0 & 3 & \cdots & 0 & -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -2 & 0 & 0 & \cdots & 3 & -1 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Divide every row except the first one by 3 to get

$$\begin{bmatrix} 1 & -1 & -1 & \cdots & -1 & 1 & 0 & 0 & \cdots & 0 \\ -\frac{2}{3} & 1 & 0 & \cdots & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & \cdots & 0 \\ -\frac{2}{3} & 0 & 1 & \cdots & 0 & -\frac{1}{3} & 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{2}{3} & 0 & 0 & \cdots & 1 & -\frac{1}{3} & 0 & 0 & \cdots & \frac{1}{3} \end{bmatrix}.$$

Add every row except the first one to the first one and then divide it by the first entry to get

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & \frac{4-n}{5-2n} & \frac{1}{5-2n} & \frac{1}{5-2n} & \cdots & \frac{1}{5-2n} \\ -\frac{2}{3} & 1 & 0 & \cdots & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & \cdots & 0 \\ -\frac{2}{3} & 0 & 1 & \cdots & 0 & -\frac{1}{3} & 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{2}{3} & 0 & 0 & \cdots & 1 & -\frac{1}{3} & 0 & 0 & \cdots & \frac{1}{3} \end{bmatrix}.$$

Finally, add the first row multiplied by $\frac{2}{3}$ to others to get the inverse of M is that

$$M^{-1} = \begin{bmatrix} \frac{4-n}{5-2n} & \frac{1}{5-2n} & \frac{1}{5-2n} & \cdots & \frac{1}{5-2n} \\ \frac{1}{5-2n} & \frac{7-2n}{3(5-2n)} & \frac{2}{3(5-2n)} & \cdots & \frac{2}{3(5-2n)} \\ \frac{1}{5-2n} & \frac{2}{3(5-2n)} & \frac{7-2n}{3(5-2n)} & \cdots & \frac{2}{3(5-2n)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{5-2n} & \frac{2}{3(5-2n)} & \frac{2}{3(5-2n)} & \cdots & \frac{7-2n}{3(5-2n)} \end{bmatrix}.$$

or

$$M^{-1} = \frac{1}{5-2n} \begin{bmatrix} 4-n & 1 & 1 & \cdots & 1 \\ 1 & \frac{7-2n}{3} & \frac{2}{3} & \cdots & \frac{2}{3} \\ 1 & \frac{2}{3} & \frac{7-2n}{3} & \cdots & \frac{2}{3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{2}{3} & \frac{2}{3} & \cdots & \frac{7-2n}{3} \end{bmatrix}.$$

3 Find the inverse of

$$M = \begin{bmatrix} \xi^{1\cdot 1} & \xi^{1\cdot 2} & \cdots & \xi^{1\cdot n} \\ \xi^{2\cdot 1} & \xi^{2\cdot 2} & \cdots & \xi^{2\cdot n} \\ \vdots & \vdots & \ddots & \vdots \\ \xi^{n\cdot 1} & \xi^{n\cdot 2} & \cdots & \xi^{n\cdot n} \end{bmatrix}$$

where $\xi = \exp\left(\frac{2\pi i}{n}\right)$ is the unit root of $z^n - 1 = 0$.

Solution.

Consider the square of M, whose (i,j)-entry is

$$\sum_{k=1}^{n} \xi^{ik+kj} = \sum_{k=1}^{n} \left(\xi^{i+j}\right)^k.$$

When $n \mid i+j$, the value will be n, otherwise, the value will be $= \xi^{i+j}\frac{\left(\xi^{i+j}\right)^n - 1}{\xi^{i+j}-1} = 0$, so

$$M^2 = n \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}.$$

Hence,

$$M^{-1} = M\left(M^2\right)^{-1} = n\begin{bmatrix} \xi^{(n-1)\cdot 1} & \xi^{(n-1)\cdot 2} & \cdots & 1 \\ \xi^{(n-2)\cdot 1} & \xi^{(n-2)\cdot 2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

4 Find the inverse of $\begin{bmatrix} B & A \\ O & B \end{bmatrix}$ **where**

$$A = \begin{bmatrix} a & a & \cdots & a \\ a & a & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & a \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix}.$$

Solution.

Let $\alpha = (1, 1, \ldots, 1)^T$, then $A = a\alpha\alpha^T$, $B = \alpha\alpha^T - I$ and $\alpha^T\alpha = n$.
Suppose the inverse of B is $k\alpha\alpha^T - I$, since $\left(\alpha\alpha^T - I\right)\left(k\alpha\alpha^T - I\right) = I$, $k = \frac{1}{n-1}$.
Hence,

$$\begin{bmatrix} B & A \\ O & B \end{bmatrix}^{-1} = \begin{bmatrix} B^{-1} & -B^{-1}AB^{-1} \\ O & B^{-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{n-1}\alpha\alpha^T - I & -\frac{a}{(n-1)^2}\alpha\alpha^T \\ O & \frac{1}{n-1}\alpha\alpha^T - I \end{bmatrix}.$$

5 Find the inverse of circulant matrix

$$M = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ n & 1 & 2 & \cdots & n-1 \\ n-1 & n & 1 & \cdots & n-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 3 & 4 & \cdots & 1 \end{bmatrix}.$$

Solution.

Let

$$Z = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then $Z^n = I$, and M can be expressed as

$$M = I + 2Z + 3Z^2 + \cdots + nZ^{n-1}.$$

Let

$$N = I + Z + Z^2 + \cdots + Z^{n-1}.$$

We can compute that $M(Z-I) = nI - N$ and $N(Z-I) = O$.
Since $Z - I \mid M - \frac{n(n+1)}{2}I$, we have

$$\begin{aligned}
&M(Z-I) = nI - N \\
\implies &M(Z-I)\left(M - \frac{n(n+1)}{2}I\right) = n\left(M - \frac{n(n+1)}{2}I\right) - N\left(M - \frac{n(n+1)}{2}I\right) \\
\implies &M(Z-I)\left(M - \frac{n(n+1)}{2}I\right) = nM - \frac{n^2(n+1)}{2}I \\
\implies &M\left((Z-I)\left(M - \frac{n(n+1)}{2}I\right) - n\right) = -\frac{n^2(n+1)}{2}I
\end{aligned}$$

Therefore, the inverse of M is

$$M^{-1} = -\frac{2}{n^2(n+1)}\left((Z-I)\left(M - \frac{n(n+1)}{2}I\right) - nI\right).$$

or

$$M^{-1} = \frac{1}{n}\left(\frac{2}{n(n+1)}\left(I + Z + Z^2 + \cdots + Z^{n-1}\right) + Z - I\right).$$

6 Find the inverse of

$$M = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & \cdots & n \end{bmatrix}.$$

Solution.
Do row exchange to $\begin{bmatrix} M & I \end{bmatrix}$.
Subtract every row from the next row to get

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 & -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 1 & 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Then subtract every row from the previous row to get

$$M^{-1} = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

7 Find the inverse of

$$M = \begin{bmatrix} k^0 \cdot C_0^0 & 0 & \cdots & 0 \\ k^1 \cdot C_1^0 & k^0 \cdot C_1^1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ k^n \cdot C_0^0 & k^{n-1} \cdot C_n^1 & \cdots & k^0 \cdot C_n^n \end{bmatrix}.$$

Solution.

Let $\alpha(x) = (x^0, x^1, \ldots, x^n)$, according to binomial theorem, $M\alpha(x) = \alpha(x+k)$, so we can easily get that $M^{-1}\alpha(x) = \alpha(x-k)$, that is to say,

$$M^{-1} = \begin{bmatrix} (-k)^0 \cdot C_0^0 & 0 & \cdots & 0 \\ (-k)^1 \cdot C_1^0 & (-k)^0 \cdot C_1^1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (-k)^n \cdot C_0^0 & (-k)^{n-1} \cdot C_n^1 & \cdots & (-k)^0 \cdot C_n^n \end{bmatrix}.$$

8 Find the inverse of tridiagonal matrix

$$M = \begin{bmatrix} 2\cos x & 1 & 0 & \cdots & 0 \\ 1 & 2\cos x & 1 & \cdots & 0 \\ 0 & 1 & 2\cos x & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2\cos x \end{bmatrix}.$$

Solution.

We claim that the (i,j)-entry of M^{-1} is

$$\begin{cases} (-1)^{i+j}\frac{\sin ix \sin(n+1-j)x}{\sin(n+1)x\sin x}, & i \le j, \\ (-1)^{i+j}\frac{\sin jx \sin(n+1-i)x}{\sin(n+1)x\sin x}, & i > j. \end{cases}$$

It can be easily checked that

$$\begin{aligned} &(-1)^{i+i-1}\frac{\sin(i-1)x\sin(n+1-i)x}{\sin(n+1)x\sin x} + (-1)^{i+i}\frac{\sin ix \sin(n+1-i)x}{\sin(n+1)x\sin x}2\cos x \\ &+ (-1)^{i+i+1}\frac{\sin ix\sin(n-i)x}{\sin(n+1)x\sin x} \\ =&\frac{\cos ix \sin(n+1-i)x + \cos(n+1-i)x\sin ix}{\sin(n+1)x} \\ =&1, \end{aligned}$$

and when $j \neq i$,

$$\begin{aligned} &(-1)^{i+j-1}\frac{\sin ix\sin(n+2-j)x}{\sin(n+1)x\sin x} + (-1)^{i+j}\frac{\sin ix\sin(n+1-j)x}{\sin(n+1)x\sin x}2\cos x \\ &+ (-1)^{i+j+1}\frac{\sin ix\sin(n-j)x}{\sin(n+1)x\sin x} \\ =&(-1)^{i+j}\frac{\sin ix}{\sin(n+1)x\sin x}\left(2\cos x\sin(n+1-j)x - \sin(n+2-j)x - \sin(n-j)x\right) \\ =&0. \end{aligned}$$

That is what needs to prove.

9 Find the inverse of Hilbert matrix

$$M = \begin{bmatrix} \frac{1}{1+1} & \frac{1}{1+2} & \cdots & \frac{1}{1+n} \\ \frac{1}{2+1} & \frac{1}{2+2} & \cdots & \frac{1}{2+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{n+n} \end{bmatrix}.$$

Solution.

We first calculate the determinant of

$$N(n) = \begin{bmatrix} \frac{1}{a_1+b_1} & \frac{1}{a_1+b_2} & \cdots & \frac{1}{a_1+b_n} \\ \frac{1}{a_2+b_1} & \frac{1}{a_2+b_2} & \cdots & \frac{1}{a_2+b_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_n+b_1} & \frac{1}{a_n+b_2} & \cdots & \frac{1}{a_n+b_n} \end{bmatrix}.$$

Since

$$\frac{1}{a_i+b_j} - \frac{1}{a_n+b_j} = \frac{a_n - a_i}{(a_i+b_j)(a_n+b_j)},$$

we have

$$\det N(n) = (a_n+b_n)\frac{\prod\limits_{j=1}^{n-1}(a_n-a_j)(b_n-b_j)}{\prod\limits_{j=1}^{n}(a_n+b_j)(a_j+b_n)}\det N(n-1).$$

Hence,

$$\det N(n) = \frac{\prod\limits_{1\le i<j\le n}(a_j-a_i)(b_j-b_i)}{\prod\limits_{i=1}^{n}\prod\limits_{j=i}^{n}(a_i+b_j)}.$$

Then

$$\left(M^{-1}\right)_{ij} = (-1)^{i+j} C_n^i C_{n+i}^i C_n^j C_{n+i}^j \frac{ij}{i+j}.$$

10 Find the inverse of Vandermonde matrix

$$M = \begin{bmatrix} x_1^1 & x_1^2 & \cdots & x_1^n \\ x_2^1 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \ddots & \vdots \\ x_n^1 & x_n^2 & \cdots & x_n^n \end{bmatrix}.$$

Solution. We can use row exchange to get

$$\left(M^{-1}\right)_{ij} = (-1)^{j-1}\frac{\sigma_{n-j}}{x_i\prod\limits_{k\neq i}(x_i-x_k)}$$

where

$$\sigma_0 = 1,$$

$$\sigma_1 = x_1 + x_2 + \cdots + x_n,$$

$$\cdots,$$

$$\sigma_n = x_1 x_2 \cdots x_n.$$

2 About vector spaces

1 Prove that any matrix $A \in \mathbb{F}^{m\times n}$ can be expressed as $A = LS\tilde{I}DU$ where L is lower triangular, S is a permutation matrix, $\tilde{I}$ is the Hermitian form of A, D is diagonal and U is upper triangular.

Proof. We can use the method of Gaussian Elimination.
First, for every non-zero column from left to right, find the first non-zero component that is the first non-zero entry in its row, reduce other free components in the column with row transformation (but not permutation) to zero, namely get $QA = R$ where Q is lower triangular and R is like the row echelon form of A.
Then, use permutation matrix P to exchange R's rows to the form $\begin{bmatrix} D_r & F \\ O & O \end{bmatrix}$ where D_r is diagonal.
Let $D = \begin{bmatrix} D_r & O \\ O & I_{n-r} \end{bmatrix}$, we have

$$\begin{bmatrix} D_r & F \\ O & O \end{bmatrix} = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} \begin{bmatrix} D_r & O \\ O & I_{n-r} \end{bmatrix} \begin{bmatrix} I_r & F_0 \\ O & I_{n-r} \end{bmatrix}.$$

Let $S = P^{-1}$, $L = Q^{-1}$, $U = \begin{bmatrix} I_r & F_0 \\ O & I_{n-r} \end{bmatrix}$, then we can factorize A as $LS\tilde{I}DU$. □

2 Suppose V is a 2024-dimensional vector space over field $\mathbb{F}$, try to find a set S of 2025 vectors in V, such that any 2024 vectors in S are linearly independent.
Solution.
Take $e_1, e_2, \ldots, e_{2024}$ that is a basis of V, let

$$S := \{e_1, e_2, \ldots, e_{2024}, e_1 + e_2 + \cdots + e_{2024}\}.$$

If any 2024 vectors in S are linearly dependent, we can immediately conclude that $e_1, e_2, \ldots, e_{2024}$ are linearly dependent, which is contradict to the definition of basis.

3 Suppose V is an n-dimensional vector space over number field $\mathbb{F}$, try to find an infinite set S of vectors in V, such that any n vectors in S are linearly independent. What if $\mathbb{F}$ is finite?
Solution.
When $\mathbb{F}$ is a number field, consider matrix

$$M = \begin{bmatrix} 1^1 & 1^2 & 1^3 & \cdots & 1^n \\ 2^1 & 2^2 & 2^3 & \cdots & 2^n \\ 3^1 & 3^2 & 3^3 & \cdots & 3^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix}.$$

S is defined to contain all the row vectors of M, such that any n vectors in S form a Vandermonde matrix, which is invertible, namely the vectors are linearly independent.
When $\mathbb{F}$ is a finite field, there are only finite vectors in V, so we cannot find an infinite set S.

4 Suppose V is an n-dimensional vector space over number field $\mathbb{F}$, prove that for any m subspaces $U_i \subset V$, $U_i \neq V$, there exists $\bigcup\limits_{i=1}^{m} U_i \neq V$. When $\mathbb{F}$ is not a number field, try to give a counter example.

Proof. We will give the proof by induction.
When $m = 1$, we have $U_1 \neq V$.
Suppose $\bigcup\limits_{i=1}^{k} U_i \neq V$, take $\alpha \in V \setminus \bigcup\limits_{i=1}^{k} U_i$. If $\alpha \notin U_{k+1}$, $\bigcup\limits_{i=1}^{k+1} U_i \neq V$, otherwise, take $\beta \in V \setminus \bigcup\limits_{i=1}^{k-1} U_i \cup U_{k+1}$. If $\beta \notin U_k$, $\bigcup\limits_{i=1}^{k+1} U_i \neq V$, otherwise, consider infinite set (as $\mathbb{F}$ is a number field)

$$S := \{\alpha + \beta, \alpha + 2\beta, \alpha + 3\beta, \dots\},$$

at most one element in S belongs to U_i, or we have $\alpha, \beta \in U_i$. Since k is finite, we can choose $\alpha + t\beta \in S$ such that $\alpha + t\beta \notin \bigcup\limits_{i=1}^{k+1} U_i$, so $\bigcup\limits_{i=1}^{k+1} U_i \neq V$.
When $\mathbb{F}$ is not a number field, we can choose $\mathbb{F} = \mathbb{F}_2$ and $V = \mathbb{F}^2$, let

$$U_1 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad U_2 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}, \quad U_3 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

We have $U_1 \cup U_2 \cup U_3 = V$. □

5 Suppose V is an n-dimensional vector space over any field $\mathbb{F}$, $U_1 \neq V$ and $U_2 \neq V$ are two subspaces of V, prove that $U_1 \cup U_2 \neq V$.

Proof. If $U_1 \subset U_2$ or $U_2 \subset U_1$, we have $U_1 \cup U_2 = U_2 \neq V$ or $U_1 \cup U_2 = U_1 \neq V$.
Otherwise, there exists $\alpha \in U_1$ such that $\alpha \notin U_2$ and $\beta \in U_2$ such that $\beta \notin U_1$.
Then $\alpha + \beta \notin U_1 \cup U_2$, so $U_1 \cup U_2 \neq V$. □

6(Refer to former example) Suppose U_1, U_2, U_3 are three subspaces of V, prove that

$$(U_1 + U_2) \cap U_3 \supset (U_1 \cap U_3) + (U_2 \cap U_3),$$

$$U_1 \cap U_2 + U_3 \subset (U_1 + U_3) \cap (U_2 + U_3),$$

and falsify the opposite side.

Proof. $\forall u_1 + u_2 \in (U_1 \cap U_3) + (U_2 \cap U_3)$, we have $u_1 \in U_1$, $u_2 \in U_2$, and $u_1, u_2 \in U_3$, then $u_1 + u_2 \in U_1 + U_2$ and $u_1 + u_2 \in U_3$, so $u_1 + u_2 \in (U_1 + U_2) \cap U_3$, which means $(U_1 + U_2) \cap U_3 \supset (U_1 \cap U_3) + (U_2 \cap U_3)$.
$\forall v + u \in U_1 \cap U_2 + U_3$, we have $v + u \in U_1 + U_3$ and $v + u \in U_2 + U_3$, so $U_1 \cap U_2 + U_3 \subset$

$(U_1 + U_3) \cap (U_2 + U_3)$.
Take

$$U_1 = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \mid x \in \mathbb{R} \right\}.$$

$$U_2 = \left\{ \begin{bmatrix} 0 \\ x \end{bmatrix} \mid x \in \mathbb{R} \right\}.$$

$$U_3 = \left\{ \begin{bmatrix} x \\ x \end{bmatrix} \mid x \in \mathbb{R} \right\}.$$

as a counter example. □

7(Refer to former example) Prove the modular lattice structure of subspace, that is to say, if U_- is a subspace of U_+, we have

$$(U_- + U_0) \cap U_+ = U_- + (U_0 \cap U_+) .$$

Proof. We can use the conclusion in 6.

$$U_- + (U_0 \cap U_+) \subset (U_- + U_0) \cap (U_- + U_+) .$$

As U_- is a subspace of U_+, we have $U_- + U_+ = U_+$, so $U_- + (U_0 \cap U_+) \subset (U_- + U_0) \cap U_+$.
$\forall u+v \in (U_- + U_0) \cap U_+$, we have $u \in U_-$, $v \in U_0$ and $u+v \in U_+$, since U_- is a subspace of U_+, we have $u \in U_+$,$-u \in U_+$, so $v = u + v - u \in U_+$, then $v \in U_0 \cap U_+$, $u + v \in U_- + (U_0 \cap U_+)$, that is to say, $(U_- + U_0) \cap U_+ \subset U_- + (U_0 \cap U_+)$.
To sum up, we have $(U_- + U_0) \cap U_+ = U_- + (U_0 \cap U_+)$. □

8 Suppose U_1, U_2, U_3 are three subspaces of V, prove that

$$(U_3 \cap U_1 + U_2) \cap U_3 = U_3 \cap (U_1 + U_2 \cap U_3) ,$$

$$(U_3 + U_1) \cap U_2 + U_3 = U_3 + (U_1 \cap U_2) + U_3.$$

Proof. We only need to prove that

$$(U_3 \cap U_1 + U_2) \cap U_3 = U_1 \cap U_3 + U_2 \cap U_3,$$

$$(U_3 + U_1) \cap U_2 + U_3 = (U_1 + U_3) \cap (U_2 + U_3) .$$

Since $U_3 \cap U_1 \subset U_3$, by modular lattice structure, we have

$$(U_3 \cap U_1 + U_2) \cap U_3 = U_3 \cap U_1 + U_2 \cap U_3.$$

And since $U_3 \subset U_3 + U_1$, by modular lattice structure, we have

$$U_3 + U_2 \cap (U_3 + U_1) = (U_3 + U_2) \cap (U_3 + U_1) .$$

□

9 Let S be a subset of vector space V, prove that $\bigcap\limits_{S\subset U\subset V} U$ is the smallest subspace that contains S, which will be defined as $\operatorname{span}(S)$.

Proof. Every condition of vector space that needs to be satisfied in $\bigcap\limits_{S\subset U\subset V} U$ is satisfied in every U, so it is naturally satisfied in $\bigcap\limits_{S\subset U\subset V} U$.
Since for any $S \subset U_0 \subset V$, we have $\bigcap\limits_{S\subset U\subset V} U \subset U_0$, so it is the smallest one. □

10 Let S_1 and S_2 are two maximal linearly independent sets of n-dimensional vector space V, prove that they have the same cardinality, which will be defined as $\dim(V)$, and

$$\operatorname{rank} := \dim \circ \operatorname{span}.$$

Proof. Take

$$S_1 = \{u_1, u_2, \ldots, u_r\},$$

$$S_2 = \{v_1, v_2, \ldots, v_s\}$$

are linearly independent respectively.
As S_2 can be a basis of V, for any u_i, there exists $a_{i1}, a_{i2}, \ldots, a_{is}$ such that

$$u_i = \sum_{j=1}^{s} a_{ij} v_j.$$

Then

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_r \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_s \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{sr} \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_s \end{bmatrix} A.$$

Consider equation

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{bmatrix} = 0.$$

On one side, since S_1 can be a basis of V, x has only zero solution.
On the other side, Ax also has only zero solution.
If $r > s$, A has free columns, then Ax has non-zero solutions, which is contradict to $x = 0$.
Similarly, $r < s$ is also impossible, so $r = s$. □

11 Let S_1 and S_2 be two subsets of vector space V. Prove that if $S_1 \subset S_2$, then $\operatorname{span}(S_1) \subset \operatorname{span}(S_2)$ and that if $\operatorname{span}(S_1) \subset \operatorname{span}(S_2)$, then $\operatorname{rank}(S_1) \le \operatorname{rank}(S_2)$.

Proof. If $S_1 \subset S_2$, $\forall x \in \text{span}(S_1)$, let $\{v_1, v_2, \dots, v_r\}$ be a basis of $\text{span}(S_1)$, then there exists $(a_1, a_2, \dots, a_r)$, such that

$$x = \sum_{i=1}^{r} a_i v_i.$$

Since $v_i \in S_1$, $S_1 \subset S_2$, then $v_i \in S_2$, so $x \in \text{span}(S_2)$, that is to say, $\text{span}(S_1) \subset \text{span}(S_2)$.
If $\text{span}(S_1) \subset \text{span}(S_2)$, let $\{v_1, v_2, \dots, v_r\}$ be a basis of $\text{span}(S_1)$, we have $\{v_1, v_2, \dots, v_r\} \subset \text{span}(S_2)$, so $\text{rank}(S_1) \leq \text{rank}(S_2)$. □

12 Let S_1 and S_2 be two subsets of vector space V. Prove that

1. **If $S_1 \subset S_2$, then $\text{rank}(S_1) = \text{rank}(S_2)$ if and only if $\text{span}(S_1) = \text{span}(S_2)$;**

2. **If $S_1 = S_2$, then $\text{span}(S_1) = \text{span}(S_2)$, if $\text{span}(S_1) = \text{span}(S_2)$, then $\text{rank}(S_1) = \text{rank}(S_2)$;**

3. **The other side of 2 is incorrect.**

Proof.

1. If $\text{span}(S_1) = \text{span}(S_2)$, since dim is a mapping, $\text{rank}(S_1) = \text{rank}(S_2)$.
   If $\text{span}(S_1) \neq \text{span}(S_2)$, then let $\{v_1, v_2, \dots, v_r\}$ be a maximum linearly independent set of S_1, there exists $v_{r+1} \in \text{span}(S_2) \setminus \text{span}(S_1)$, which are linearly independent, so $\text{rank}(S_1) < r + 1 < \text{rank}(S_2)$.

2. Since span and dim are mappings, this statement is correct.

3. For the former one,

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad S_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$$

   can be a counter example.
   For the latter one,

$$S_1 = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \mid x \in \mathbb{F} \right\}, \quad S_2 = \left\{ \begin{bmatrix} 0 \\ x \end{bmatrix} \mid x \in \mathbb{F} \right\}$$

   can be a counter example.

□

13 What will $\text{span}(S)$ and $\text{rank}(S)$ be if $S = \emptyset$ or $S = V$ where V is an n-dimensional vector space?

Solution.
When $S = \emptyset$, $\text{span}(S) = \{0\}$, $\text{rank}(S) = 0$.
When $S = V$, $\text{span}(S) = V$, $\text{rank}(S) = n$.

14 Given $S_1 \subset S_2 \subset V$ are two sets where V is an n-dimensional vector space, prove that

$$\mathrm{span}(S_1 \cap S_2) = \mathrm{span}(S_1) \cap \mathrm{span}(S_2),$$

$$\mathrm{span}(S_1 \cup S_2) = \mathrm{span}(S_1) + \mathrm{span}(S_2).$$

Given $U_1 \subset U_2 \subset V$ are two subspaces, prove that

$$\mathrm{rank}(U_1 \cap U_2) = \min(\mathrm{rank}(U_1), \mathrm{rank}(U_2)),$$

$$\mathrm{rank}(U_1 \cup U_2) = \max(\mathrm{rank}(U_1), \mathrm{rank}(U_2)).$$

Proof. $S_1 \subset S_2 \implies \mathrm{span}(S_1) \subset \mathrm{span}(S_2)$, so $\mathrm{span}(S_1 \cap S_2) = \mathrm{span}(S_1) = \mathrm{span}(S_1) \cap \mathrm{span}(S_2)$.
Others are similar. □

15 Given U_1 and U_2 are two subspaces of V, prove that

$$\mathrm{rank}(U_1 \cap U_2) \leq \min(\mathrm{rank}(U_1), \mathrm{rank}(U_2)),$$

$$\max(\mathrm{rank}(U_1), \mathrm{rank}(U_2)) \leq \mathrm{rank}(U_1 + U_2).$$

Proof. By 11, we have $\mathrm{rank}(U_1 \cap U_2) \leq \mathrm{rank}(U_1)$ and $\mathrm{rank}(U_1 \cap U_2) \leq \mathrm{rank}(U_2)$.
Similarly, $\mathrm{rank}(U_1 + U_2) \geq \mathrm{rank}(U_1)$ and $\mathrm{rank}(U_1 + U_2) \geq \mathrm{rank}(U_2)$. □

16 Given S_1 and S_2 are two subsets of V, prove that

$$\mathrm{span}(S_1 \cap S_2) \subset \mathrm{span}(S_1) \cap \mathrm{span}(S_2)),$$

$$\mathrm{span}(S_1) + \mathrm{span}(S_2) = \mathrm{span}(S_1 \cup S_2).$$

Proof. By 11, we have $\mathrm{span}(S_1 \cap S_2) \subset \mathrm{span}(S_1)$ and $\mathrm{span}(S_1 \cap S_2) \subset \mathrm{span}(S_2)$.
Similarly, $\mathrm{span}(S_1) \subset \mathrm{span}(S_1 \cup S_2)$ and $\mathrm{span}(S_2) \subset \mathrm{span}(S_1 \cup S_2)$.
$\forall x \in \mathrm{span}(S_1 \cup S_2)$, we can express it as $x = \sum a_i v_i$ where $v_i \in S_1$ or $v_i \in S_2$, then let us are vs in S_1, ws are other vs, then $x = \sum a_i u_i + x = \sum a_i w_i \in \mathrm{span}(S_1) + \mathrm{span}(S_2)$. □

17 Suppose U_1, U_2, U_3 are three subspaces of V, prove that

$$(U_1 + U_2) \cap U_3 \supset (U_1 \cap U_3) + (U_2 \cap U_3),$$

$$U_1 \cap U_2 + U_3 \subset (U_1 + U_3) \cap (U_2 + U_3),$$

and falsify the opposite side.

Proof. Refer to 6. □

18 Given $l, m, n \in \mathbb{N}$, prove that

$$\max(\min(l,m),\min(l,n)) = \min(l,\max(m,n)),$$

$$\min(\max(l,m),\max(l,n)) = \max(l,\min(m,n)).$$

Proof. $\max(\min(l,m),\min(l,n)) \leq l, \max(m,n)$, so $\max(\min(l,m),\min(l,n)) \leq \min(l,\max(m,n))$.
$\max(\min(l,m),\min(l,n)) \geq \min(l,m),\min(l,n)$, so $\max(\min(l,m),\min(l,n)) \geq \min(l,\max(m,n))$.
So $\max(\min(l,m),\min(l,n)) = \min(l,\max(m,n))$.
The other one is similar. □

19 Prove that vector space V is infinite dimensional if and only if there exists an infinite set $S \subset V$ such that vectors in S are linearly independent.

Proof. (Sufficiency) If V is finite dimensional, let it be n-dimensional. Take $(v_1, v_2, \ldots, v_{n+1})$ are vectors in S. Since they are linearly independent, this is a contradiction.
(Necessity) If V is infinite dimensional, any finite set of vectors cannot be its basis, that is to say, if we have found $(v_1, v_2, \ldots, v_n)$ are linearly independent, there still exists v_{n+1} that is linearly independent of $(v_1, v_2, \ldots, v_n)$, so vectors in S can be found one by one. □

20 Prove that the space of formal power series

$$\mathbb{F}[\![x]\!] := \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in \mathbb{F} \right\}$$

is not a countable dimensional vector space.

Proof. We can complete proof by contradiction.
Suppose $\mathbb{F}[\![x]\!]$ is a countable dimensional vector space, then there exists

$$S := \{f_1(x), f_2(x), \ldots\} \subset \mathbb{F}[\![x]\!]$$

such that $\operatorname{span}(S) = \mathbb{F}[\![x]\!]$.
Let

$$f_k(x) = \sum_{i=0}^{\infty} a_{ki} x^i$$

and

$$S_n := \left\{ v_k \mid v_k = \begin{bmatrix} a_{k0} \\ a_{k1} \\ \vdots \\ a_{kn} \end{bmatrix}, k = 1, 2, \ldots, n \right\}.$$

We can define a sequence $\{t_n\}$ where $t_1 = 1$. Suppose $t_1, t_2, \ldots, t_n$ have been defined, since S_n has n $(n+1)$-dimensional vectors, which cannot fill the whole $(n+1)$-dimensional vector space, we can choose t_{n+1} such that $u_{n+1} = (t_1, t_2, \ldots, t_n, t_{n+1})^T$ cannot be expressed as linear combination of vectors in S_n.

Let

$$f(x) = \sum_{i=0}^{\infty} t_{i+1} x^i.$$

Then for any finite $f_k(x) \in \mathbb{F}[\![x]\!]$, let n be the largest index, since u_{n+1} cannot be expressed as linear combination of vectors in S_n, we can conclude that $f(x)$ cannot be expressed as linear combination of $f_k(x)$s, that is to say, $f(x)$ is not in $\operatorname{span}(S)$, so $\operatorname{span}(S) \neq V$. $\square$