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We dedicate this book to our wives
Louise Enochs and Claudine Jenda.

Preface

The purpose of this second volume is to give the reader some feeling for problems and proofs concerning complexes. So we have included some basic results. These are probably well known, but it may be hard to find their proofs all in one place. We have also tried to give a sampling of some of the new developments and the associated tools in the study of complexes. Then we hope we will have encouraged the readers to go on to learn about more advanced topics such as derived categories and dualizing complexes. Bibliographical notes at the end of the volume describe references for extra reading.

We would like to again thank Mrs. Rosie Torbert for continuing to prepare the manuscripts for us.

Lexington/Auburn, April 2011 Edgar E. Enochs, Overtoun M. G. Jenda

Nomenclature

$^{\perp}\mathcal{A}$	All complexes B such that $\mathrm{Ext}^1(B, A) = 0$ for all $A \in \mathcal{A}$, page 37
$\mathcal{A}^{\perp}$	All complexes C such that $\mathrm{Ext}^1(A, C) = 0$ for all $A \in \mathcal{A}$, page 37
$B(C)$	A subcomplex of C where $B(C)_n = \mathrm{Im}(d_{n+1})$, page 3
C–E injective complex	Cartan–Eilenberg injective complex, page 88
C–E projective complex	Cartan–Eilenberg projective complex, page 83
$\lvert C\rvert$	Cardinality of a complex C, page 1
$C(\mathcal{A})$	The class of complexes A such that $A_n \in \mathcal{A}$ for each $n \in \mathbb{Z}$ where $\mathcal{A}$ is a class of objects of R-Mod, page 66
(C, d)	A complex of left R-modules with differential d of C, page 1
$C(f)$	The complex such that $C(f)_n = D_n \oplus C_{n-1}$ and where $f : C \to D$ is a morphism of complexes with the differential defined by $d_n^{C(f)}(y, x) = (d_n^D(y) + f_{n-1}(x), -d_{n-1}^C(x))$, page 24
$C(R\text{-Mod})$	Additive category of a complex of left R-modules, page 2
$\mathcal{E}xt(M, N)$	Set of all equivalence classes of short exact sequences $0 \to N \to U \to M \to 0$, page 20
$\mathrm{Ext}^n(C, D)$	n^{th} homology group of complexes C and D of left R-modules, page 14
$\mathrm{Filt}(\mathcal{S})$	Class of complexes that have an $\mathcal{S}$-filtration, page 38
$\mathrm{Gr}(R\text{-Mod})$	The category whose objects are families $(M_n)_{n\in\mathbb{Z}}$ of left R-modules indexed by the set $\mathbb{Z}$ of integers, page 27
$H(C)$	Homology of a complex C where $H(C)_n = Z(C)_n / B(C)_n$, page 3
$\mathcal{H}(M)$	A Hill class of submodules of M, page 70

$\mathcal{H}om(C, D)$	Graded abelian group of morphisms, page 34
$\mathrm{Hom}_{C(R\text{-Mod})}(C, D)$	Abelian group of morphisms $f : C \to D$, page 2
$K(\mathcal{A})$	The full subcategory of $K(R\text{-Mod})$ having the same objects as the full subcategory $\mathcal{A}$ of $C(R\text{-Mod})$, page 51
$K(R\text{-Mod})$	The category whose objects are complexes of left R-modules and whose morphisms $C \to D$ are equivalence classes up to homotopy, page 29
$\bar{M}$	The complex $\cdots \to 0 \to M \xrightarrow{1} M \to 0 \to \cdots$ for a left R-module M, page 4
$\underline{M}$	The complex $\cdots \to 0 \to M \to 0 \to \cdots$ for a left R-module M, page 4
$S(f)$	If $f : C \to D$ is a morphism, then $S(f)$ is a morphism $S(C) \to S(D)$ where $S(f)_n(x) = f_{n-1}(x)$ for $x \in C_{n-1}$, page 3
$S^k(C)$	k^{th} suspension of a complex C where $S^k(C)_n = C_{n-k}$. $S^1(C)$ is denoted simply as $S(C)$, page 3
$Z(C)$	A subcomplex of C where $Z(C)_n = \mathrm{Ker}(d_n)$ for each n, page 3

Contents

Chapter 1
Complexes of Modules

In this chapter, we will consider categories of complexes of modules and will give some of their basic properties. We characterize the projective and injective objects in these categories and prove the complex version of the Baer criterion.

1.1 Definitions and Basic Constructions

In what follows R will be a ring.

Definition 1.1.1. By a *complex* of left R-modules we mean a pair $((C_n)_{n\in\mathbb{Z}}, (d_n)_{n\in\mathbb{Z}})$ where each C_n is a left R-module and where $d_n : C_n \to C_{n-1}$ is a linear map such that $d_{n-1} \circ d_n = 0$ for all $n \in \mathbb{Z}$. We usually abbreviate and denote $((C_n)_{n\in Z}, (d_n)_{n\in Z})$ as (C, d) or simply as C (with d understood). We call d the *differential* of C and the modules C_n the *terms* of C.

Remark 1.1.2. If it is convenient to use superscripts instead of subscripts we let $C^n = C_{-n}$ and $d^n = d_{-n}$. So we have $d^n : C^n \to C^{n+1}$.

However, we will frequently use superscripts to distinguish complexes. So we will let $(C^i)_{i\in I}$ denote a family of complexes indexed by $i \in I$.

It is convenient to assume that for a complex C we have $C_n \cap C_m = \emptyset$ when $n \neq m$. So then we take $x \in C$ to mean $x \in \bigcup_{n\in\mathbb{Z}} C_n$ and we write $\deg(x) = n$ if $x \in C_n$. We note that in actual practice we may have $C_n \cap C_m \neq \emptyset$ with $n \neq m$. The *cardinality* (denoted $|C|$) of complex C is defined to be $\sum_{n\in\mathbb{Z}} |C_n|$, i.e. $|\bigcup_{n\in\mathbb{Z}} C_n|$.

Sometimes we need to distinguish the differentials of various complexes. To do so we often use the obvious conventions. So for example, we might let d and d' be the differentials of C and C'. Another convention is to use d^C to indicate the differential of the complex C.

Many results for modules easily carry over to complexes. So we will often limit ourselves to stating results without proofs. But we suggest that the reader who is less familiar with complexes check these out.

In the rest of the chapter, by "complex" we mean a complex of left R-modules for some ring R. Similarly "module" will mean a left R-module. We will let R-Mod denote the category of left R-modules.

Definition 1.1.3. Given complexes C' and C of left R-modules, by a *morphism* $f : C' \to C$ we mean a family $(f_n)_{n \in \mathbb{Z}}$ of linear maps $f_n : C'_n - C_n$ such that $d_n \circ f_n = f_{n-1} \circ d'_n$ for all $n \in \mathbb{Z}$.

So the above means that we have a commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C'_{n+1} & \xrightarrow{d'_{n+1}} & C'_n & \xrightarrow{d'_n} & C'_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \longrightarrow & \cdots
\end{array}$$

It is easy to check that with this definition we get an *additive category*. This category will be denoted $C(R\text{-Mod})$. So $\operatorname{Hom}_{C(R\text{-Mod})}(C, D)$ will denote the abelian group of morphisms $f : C \to D$.

The notation $C \in C(R\text{-Mod})$ will mean that C is a complex of left R-modules.

If R is a commutative ring, $r \in R$ and $f : C \to D$ is a morphism in $C(R\text{-Mod})$, then $rf : C \to D$ defined by $(rf)(x) = r(f(x))$ for $x \in C$ is a morphism in $C(R\text{-Mod})$. So we see that when R is a commutative ring, $\operatorname{Hom}_{C(R\text{-Mod})}(C, D)$ can be made into an R-module.

Definition 1.1.4. If $C = ((C_n), (d_n))$ (we omit the $n \in \mathbb{Z}$) is a complex, then $C' = ((C'_n), (d'_n))$ is said to be a *subcomplex* of C if C'_n is a submodule of C_n for each n and if d_n agrees with d'_n on C'_n.

If C' is a subcomplex of C, we write $C' \subset C$. If $(C^i)_{i \in I}$ is a family of subcomplexes of C, there are obvious subcomplexes of C that will be represented by the symbols $\bigcap_{i \in I} C^i$ and $\sum_{i \in I} C^i$.

To say that C is the *direct sum* of a family $(C^i)_{i \in I}$ of subcomplexes will have the obvious meaning. And so then to say that a subcomplex $S \subset C$ is a *direct summand* of C will also have the obvious meaning.

If $S \subset C$ is a direct summand, then for each $n \in \mathbb{Z}$, S_n is a direct summand of C_n. However it may happen that S_n is a direct summand of C_n for each n without S being a direct summand of C.

If $S \subset C$ is a subcomplex, then for any $n \in \mathbb{Z}$, $d_n : C_n \to C_{n-1}$ induces a map $C_n/S_n \to C_{n-1}/S_{n-1}$. With these maps we get a *quotient complex* which will be denoted C/S.

If $f : C \to D$ is a morphism of complexes, then we check that $d_n(\operatorname{Ker}(f_n)) \subset \operatorname{Ker}(f_{n-1})$. This shows that we get a subcomplex of C whose n^{th} term is $\operatorname{Ker}(f_n)$. This subcomplex will be denoted $\operatorname{Ker}(f)$.

In a similar manner, we get a complex denoted $\operatorname{Im}(f)$ with $\operatorname{Im}(f) \subset D$. So then we get the complexes $\operatorname{Coker}(f) = D/\operatorname{Im}(f)$ and $\operatorname{Coim}(f) = C/\operatorname{Ker}(f)$.

If $f : C \to D$ is a morphism and S is a subcomplex of C with $S \subset \text{Ker}(f)$, then we get an induced morphism $C/S \to D$ along with the usual claims about $C/S \to D$. So finally we see that $C(R\text{-Mod})$ is an *Abelian category*. (See Enochs–Jenda [8, Section 1.3]). We say a diagram $C' \xrightarrow{f} C \xrightarrow{g} C''$ of complexes is *exact* if $\text{Im}(f) = \text{Ker}(g)$. So for each $n \in \mathbb{Z}$, $C'_n \to C_n \to C''_n$ is an exact sequence of modules. We then generalize this notion to longer sequences of complexes. So, for example, $C^1 \to C^2 \to C^3 \to C^4$ is exact if and only if $C^1 \to C^2 \to C^3$ and $C^2 \to C^3 \to C^4$ are both exact.

An exact sequence of the form $0 \to C' \to C \to C'' \to 0$ is called a *short exact sequence* of complexes.

Definition 1.1.5. By the *suspension* of a complex C we mean the complex denoted $S(C)$ where $S(C)_n = C_{n-1}$ and whose differential is $-d$ where d is the differential of C (more precisely, $d_n^{S(C)} = -d_{n-1}^C$ for any n). Then we define $S^k(C)$ for any $k \in \mathbb{Z}$ in the obvious fashion with $S^k(C)_n = C_{n-k}$. Note that $\text{Hom}_{C(R\text{-Mod})}(S(C), D) \cong \text{Hom}_{C(R\text{-Mod})}(C, S^{-1}(D))$ for any C and D.

More generally, we have

$$\text{Hom}_{C(R\text{-Mod})}(S^k(C), D) \cong \text{Hom}_{C(R\text{-Mod})}(C, S^{-k}(D))$$

for any $k \in \mathbb{Z}$.

If $f : C \to D$ is a morphism, we get a morphism $S(C) \to S(D)$ denoted $S(f)$. So $S(f)_n(x) = f_{n-1}(x)$ for $x \in C_{n-1}$. Hence S is an additive functor from $C(R\text{-Mod})$ to $S(R\text{-Mod})$. In fact it is an automorphism of the category $C(R\text{-Mod})$.

If C is a complex, we see that we have a subcomplex denoted $Z(C)$ of C where $Z(C)_n = \text{Ker}(d_n)$ for each n. Note that the differential of $Z(C)$ is 0. Similarly we define a subcomplex $B(C) \subset C$ where $B(C)_n = \text{Im}(d_{n+1})$. Since $d_n \circ d_{n+1} = 0$, we get $B(C)_n \subset Z(C)_n$ and so $B(C)$ is a subcomplex of $Z(C)$. The *quotient complex* $Z(C)/B(C)$ is denoted $H(C)$.

The elements of $Z(C)$ are called the *cycles* of C and the elements of $B(C)$ are called the *boundaries* of C. The groups $H(C)_n$ are called the *homology modules* of C.

The modules $Z(C)_n$, $B(C)_n$ and $H(C)_n$ are usually denoted $Z_n(C)$, $B_n(C)$ and $H_n(C)$, respectively. So then we have $H_n(C) = Z_n(C)/B_n(C)$. The complex C is said to be *exact* if $H(C) = 0$, or equivalently if $\text{Ker}(d_n) = \text{Im}(d_{n+1})$ for all $n \in \mathbb{Z}$.

We can regard Z, B and H as additive functors

$$C(R\text{-Mod}) \to C(R\text{-Mod})$$

where $Z(f)$, $B(f)$ and $H(f)$ for a morphism $f : C \to D$ are defined in a natural fashion.

If $f : C \to D$ is an isomorphism of complexes, then $H(f) : H(C) \to H(D)$ is also an isomorphism. The converse is not true in general.

Definition 1.1.6. A morphism $f : C \to D$ is said to be a *homology isomorphism* if $H(f) : H(C) \to H(D)$ is an isomorphism.

For a family of complexes $(C^i)_{i \in I}$ we have $Z(\bigoplus_{i \in I} C^i) = \bigoplus_{i \in I} Z(C^i)$ and $B(\bigoplus_{i \in I} C^i) = \bigoplus_{i \in I} B(C^i)$. Hence $H(\bigoplus_{i \in I} C^i) \cong \bigoplus_{i \in I} H(C^i)$. The analogous result holds for $\prod_{i \in I} C^i$. Hence we see that $\bigoplus_{i \in I} C^i$ is exact if and only if each C^i is exact and that $\prod_{i \in I} C^i$ is exact if and only if each C^i is exact.

Definition 1.1.7. We refer to Definitions 1.5.1 and 1.5.2 and Theorem 1.5.3 of Volume I. Just as for modules we can define a *direct system* $(C^i, (f_{ji}))$ (with $i, j \in I$, I a direct set) of complexes of left R-modules. Then we can form the *direct* (or *inductive*) *limit* $\varinjlim C^i$ and we get the usual universal property associated with the morphisms $C^j \to \varinjlim C^i$.

Proposition 1.1.8. *For a direct system* $(C^i, (f_{ji}))$ *in* $C(R\text{-Mod})$*, we have an isomorphism*

$$H(\varinjlim C^i) \cong \varinjlim H(C^i).$$

Proof. The maps $C^j \to \varinjlim C^i$ give maps $H(C^j) \to H(\varinjlim C^i)$. So we get a map of the limit of the direct system $(H(C^i), H(f_{ji}))$ into $\varinjlim H(C^i)$, i.e. a map

$$\varinjlim H(C^i) \to H(\varinjlim C^i).$$

It is then a simple direct limit argument to get that this map is an isomorphism. □

We note that this result gives that if each C^i in such a system is exact then so is $\varinjlim C^i$.

1.2 Complexes Formed from Modules

Definition 1.2.1. If M is a module, we let $\bar{M}$ denote the complex

$$\cdots \to 0 \to M \xrightarrow{1} M \to 0 \to \cdots$$

where the two M's are in the 1st and 0th place. We let $\underline{M}$ denote the complex

$$\cdots \to 0 \to M \to 0 \to \cdots$$

with M in the 0th place.

Note that $\underline{M}$ is a subcomplex of $\overline{M}$ and that $\overline{M}/\underline{M} = S(\underline{M})$.

If C is an arbitrary complex, then a morphism $\overline{M} \to C$ is given by a commutative diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & M & \xrightarrow{1} & M & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & C_1 & \longrightarrow & C_0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

So the linear $M \to C_0$ is determined by the linear $M \to C_1$. And conversely, any linear $M \to C_1$ gives rise to a morphism $\overline{M} \to C$. So we see that

$$\mathrm{Hom}_{C(R\text{-Mod})}(\overline{M}, C) \cong \mathrm{Hom}_R(M, C_1).$$

In a similar manner, we get that

$$\mathrm{Hom}_{C(R\text{-Mod})}(\underline{M}, C) \cong \mathrm{Hom}_R(M, Z_0(C)).$$

More generally, we have

$$\mathrm{Hom}_{C(R\text{-Mod})}(S^n(\overline{M}), C) \cong \mathrm{Hom}_{C(R\text{-Mod})}(\overline{M}, S^{-n}(C)) \cong \mathrm{Hom}_R(M, C_{n-1}).$$

Similarly,

$$\mathrm{Hom}_{C(R\text{-Mod})}(S^n(\overline{M}), C) \cong \mathrm{Hom}_R(M, Z_n(C)).$$

The isomorphism $\mathrm{Hom}_{C(R\text{-Mod})}(C, \overline{M}) \cong \mathrm{Hom}_R(C_0, M)$ can be seen to hold by considering the commutative diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 & \longrightarrow & C_{-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

Then the commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_1 & \longrightarrow & C_0 & \longrightarrow & C_{-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

gives the isomorphism

$$\mathrm{Hom}_{C(R\text{-Mod})}(C, \underline{M}) \equiv \mathrm{Hom}_R(C_0/B_0(C), M).$$

Thus

$$\mathrm{Hom}_{C(R\text{-Mod})}(S^n(C), \overline{M}) \cong \mathrm{Hom}_R(C_{-n}, M)$$

and

$$\mathrm{Hom}_{C(R\text{-Mod})}(S^n(C), \underline{M}) \cong \mathrm{Hom}_R(C_{-n}/B_{-n}(C), M).$$

1.3 Free Complexes

Definition 1.3.1. By a *graded set* X we mean a family of sets $(X_n)_{n\in\mathbb{Z}}$. If we assume (as we usually will) that $X_n \cap X_m = \emptyset$ if $n \neq m$, then we write $x \in X$ to mean $x \in \bigcup_{n\in\mathbb{Z}} X_n$ and we write $\deg(x) = n$ if $x \in X_n$ (the notation $|x| = n$ is also used).

For graded sets X and Y we define $X \cup Y$ and $X \cap Y$ in the obvious way.

If X and Y are graded sets, by a *morphism* $f : X \to Y$ of degree $p \in \mathbb{Z}$ we mean a family $(f_n)_{n\in\mathbb{Z}}$ of functions f_n where $f_n : X_n \to X_{n+p}$ for all $n \in \mathbb{Z}$. So then if $f : X \to Y$ has degree p and $g : Y \to Z$ has degree q, then $g \circ f : X \to Z$ has degree $p + q$.

So we get a category with $\mathrm{Hom}(X, Y)$ denoting the set of morphisms $X \to Y$ (of any degree). So we see that in this category, $\mathrm{Hom}(X, Y)$ has the structure of a graded set with $\mathrm{Hom}(X, Y)_p$ being the set of morphisms $f : X \to Y$ of degree p.

If X is a graded set, we define the *suspension* $S(X)$ as we did for complexes. So $S(X)_n = X_{n-1}$. And then we define $S^k(X)$ for any $k \in \mathbb{Z}$.

By the *cardinality* of a graded set X (denoted $|X|$ or $\mathrm{card}(X)$), we mean $\sum_{n\in\mathbb{Z}} |X|_n$ (or $\sum_{n\in\mathbb{Z}} \mathrm{card}(X_n)$).

"Forgetting" the obvious things we see that a complex C gives rise to a graded set. Then the differential $d : C \to C$ is a morphism of degree -1.

If X is a graded set and C a complex, $X \subset C$ will mean $X_n \subset C_n$ for all $n \in \mathbb{Z}$.

Definition 1.3.2. If X is a graded subset of the complex C, we say $S \subset C$ is the *subcomplex* generated by X if S is the intersection of all subcomplexes of C that contain X. C is said to be a *finitely generated complex* if there is a finite set $X \subset C$ that generates C.

Definition 1.3.3. A complex F is said to be a *free complex with base* B if $B \subset F$ is a graded subset of F such that for any complex C and any morphism $B \to C$ of graded sets of degree 0, there is a unique morphism $F \to C$ of complexes that agrees with the morphism $B \to C$.

We say F is *free* if it has a base.

Example 1.3.4. The complex $\overline{R} = \ \to 0 \to 0 \to R \to R \to 0 \to 0 \cdots$ is free with the base B consisting of the $1 \in R$ of degree $+1$.

Now we note that if F is free with base $B \subset F$, then for any $k \in \mathbb{Z}$, $S^k(F)$ is free with base $S^k(B)$. Also, if $(F^i)_{i \in I}$ is any family of free complexes, then $\bigoplus_{i \in I} F^i$ is also free.

Using these observations we see that we can construct a free complex F with base B such that each $|B_n|$ is some specified cardinal number.

Proposition 1.3.5. *Given any complex C, there is a free complex F and an epimorphism $F \to C$.*

Proof. It suffices to find a free F with a base B such that $|B_n| \geq |C_n|$. Then there is a degree 0 epimorphism $B \to C$ (of graded sets). So the corresponding $F \to C$ is necessarily an epimorphism. □

Definition 1.3.6. A complex C is said to be *finitely presented* if there is an exact sequence

$$Q \to P \to C \to 0$$

with Q and P finitely generated free complexes.

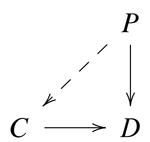
The argument for the next result is like the argument when we use modules instead of complexes.

Proposition 1.3.7. *If C is a finitely presented complex in $C(R\text{-Mod})$ and $(C^i, (f_{ji}))$ is a direct system in $C(R\text{-Mod})$, then*

$$\operatorname{Hom}(C, \varinjlim C^i) \cong \varinjlim \operatorname{Hom}(C, C^i).$$

1.4 Projective and Injective Complexes

Definition 1.4.1. A complex P is said to be *projective* if for any morphism $P \to D$ and any epimorphism $C \to D$, the diagram



can be completed to a commutative diagram by a morphism $P \to C$.

Proposition 1.4.2. *Any free complex F is projective.*

Proof. Given a base $B \subset F$ and a diagram as above we can complete the diagram

$$\begin{array}{ccc} & & B \\ & \swarrow & \downarrow \\ C & \longrightarrow & D \end{array}$$

Then find the corresponding linear $F \to C$. □

Corollary 1.4.3. *A complex P is projective if and only if it a direct summand of a free complex.*

Proof. As for modules, just complete

$$\begin{array}{ccc} & & P \\ & \swarrow & \downarrow 1 \\ F & \longrightarrow & P \end{array}$$

where F is free and $F \to P$ is an epimorphism. For the converse we just note that a direct sum of complexes is projective if and only if each of the summands is projective. □

Remark 1.4.4. Since P is then isomorphic to a direct summand of F, we get P_n is isomorphic to a direct summand of F_n. It is not hard to see that if F is free then each F_n is a free module. Hence P_n is a projective module. So this is a necessary condition on P in order that P be projective. The next example shows that the condition is not sufficient.

Example 1.4.5. $\overline{R}$ has $\underline{R}$ as a subcomplex and $\overline{R}/\underline{R} = S(\underline{R})$. Since

$$\mathrm{Hom}_{C(R\text{-Mod})}(S(\underline{R}), \overline{R}) = 0,$$

the diagram

$$\begin{array}{ccc} & & S(\underline{R}) \\ & \swarrow & \downarrow \underline{1} \\ \bar{R} & \longrightarrow & S(\underline{R}) \end{array}$$

cannot be completed to a commutative diagram. So $S(\underline{R})$ (and so also $S^{-1}(S(\underline{R})) = \underline{R}$) is not a projective complex even though each term of $S(\underline{R})$ is a projective module.

However we have the following.

Proposition 1.4.6. *If P is a projective module, then $\bar{P}$ is a projective complex.*

Proof. Given a diagram

$$\begin{array}{ccc} & & \bar{P} \\ & \swarrow & \downarrow \\ C & \longrightarrow & D \end{array}$$

with $C \to D$ an epimorphism of complexes, we use the remarks of Section 1.2 above to get the diagram

$$\begin{array}{ccc} & & P \\ & \swarrow & \downarrow \\ C_1 & \longrightarrow & D_1 \end{array}$$

where $C_1 \to D_1$ is surjective. Then since P is a projective module we get a linear $P \to C_1$ that makes the diagram commutative. And then again by Section 1.2, we see that we get a morphism $\overline{P} \to C$ that makes the original diagram commutative. □

Theorem 1.4.7. *Let P be a complex. Then the following are equivalent:*

a) *P is projective*

b) *P is a direct summand of a free complex*

c) *There is a family $(P^n)_{n\in\mathbb{Z}}$ of projective modules such that $P \cong \bigoplus_{n\in\mathbb{Z}} S^n(\overline{P^n})$*

d) *P is exact and $Z(P)$ has all its terms projective*

Furthermore, for a projective P, the family $(P^n)_{n\in\mathbb{Z}}$ in c) *above is unique up to isomorphism.*

Proof. a) $\Leftrightarrow$ b) was shown above. We argue b) $\Rightarrow$ d). If $F = P \oplus Q$ where F is free then $H(F) = H(P) \oplus H(Q)$. Since F is the direct sum of various $S^n(\overline{R})$'s and since these are exact we have F is exact. So $H(F) = 0$ and hence $H(P) = 0$. Now considering $Z(F) = Z(P) \oplus Z(Q)$ and noting that $Z(S^n(\overline{R})) = S^n(\underline{R})$, we see that $Z(F)$ has free terms. Hence $Z(P)$ has projective terms.

We now show d) $\Rightarrow$ c). By d) we have the exact $P_n \to Z_{n-1}(P) \to 0$ exact for every $n \in \mathbb{Z}$. Since $Z_{n-1}(P)$ is projective we get a section and we have $P_n \cong Z_n(P) \oplus Z_{n-1}(P)$. So P is the direct sum of the complexes

$$\cdots \to 0 \to Z_{n-1}(P) \xrightarrow{1} Z_{n-1}(P) \to 0 \to \cdots$$

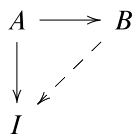
i.e. $P \cong \bigoplus_{n\in\mathbb{Z}} S^{n-1}\overline{Z_{n-1}(P)}$. This gives c).

c) $\Rightarrow$ a) follows from the observation that every $S^n(\overline{P^n})$ is projective. □

Proposition 1.4.8. *Let P be an exact complex such that each P_n $(n \in Z)$ is a projective module and such that for some n, $P_k = 0$ for $k < n$. Then P is a projective complex.*

Proof. Since $P_{n+1} \to P_n \to 0$ is exact and since P_n is projective this map has a section. So we can assume $P_{n+1} = P'_{n+1} \oplus P_n$ where $P'_{n+1} = \text{Ker}(P_{n+1} \to P_n)$. So then P is the direct sum of the complex $P' = \cdots \to P'_{n+2} \to P'_{n+1} \to 0$ and the complex $\cdots \to 0 \to P_n \to P_n \to 0 \to \cdots$. Then P' will also satisfy our hypothesis and so we can continue the procedure. Finally we see that we can appeal to c) of Theorem 1.4.7 and get that P is a projective complex. □

Definition 1.4.9. A complex I is said to be *injective* if for any morphism $A \to I$ and any monomorphism $A \to B$ of complexes, the diagram



can be completed to a commutative diagram by a morphism $B \to I$.

Before proving results for injective complexes, we make some observations. Note that an injective complex is a direct summand of any complex that contains it. Also note that if I is an injective module, then $S^n(\bar{I})$ is an injective complex. And for any complex C

$$\text{Hom}_{C(R\text{-Mod})}(C, S^n(\bar{I})) \cong \text{Hom}_R(C_n, I).$$

If we choose I so that we have an injection $C_n \to I$, then the corresponding morphism $C \to S^n(I)$ is such that $C_n \to (S^n(\bar{I}))_n = I$ is this map.

If we now choose an injective module I^n for each n in such a way that we have an injective linear map $C_n \to I^n$ and then form the corresponding morphism $C \to \overline{I^n}$ we see that $C \to \prod_{n\in\mathbb{Z}} \overline{I^n}$ is a monomorphism. Since $\prod_{n\in\mathbb{Z}} \overline{I^n}$ as a product of injective complexes is an injective complex, we see that there are enough injective complexes. But then if C itself is injective we get that C is isomorphic to a direct summand of $\prod_{n\in\mathbb{Z}} \overline{I^n}$. But now we observe that $\prod_{n\in\mathbb{Z}} S^n(\overline{I^n}) = \bigoplus_{i\in Z} S^n(\overline{I^n})$. With these observations we can get the basic results about injective complexes.

We first note that this description of the injective complexes shows that every injective complex is exact.

We now want to prove a Baer criterion for injective complexes. So we briefly recall how we get the criterion for modules. We say a module E is injective for a module M if for every submodule $S \subset M$ and every linear $S \to E$ there is an extension $M \to E$. If E is injective for M and $S \subset M$ then it is injective for S and M/S. If E is injective for M_1 and M_2 and if $S \subset M_1 \oplus M_2$ and if $f : S \to E$ is linear then we extend $S \cap M_1 \to E$ to $M_1 \to E$. But then the linear maps $S \to E$, $M_1 \to E$ agree on $S \cap M_1$ and so can be combined to give a linear map $M_1 + S \to E$. But since $M_1 \oplus 0 \subset M_1 + S$, we have $M_1 + S = M_1 \oplus T$ for some $T \subset M_2$. Extending $T \to E$ and combining with $M_1 + S = M_1 \oplus T \to E$, we get the desired $M_1 \oplus M_2 \to E$. Using this argument and Zorn's lemma we get

Proposition 1.4.10. *If E is injective for each M_i in some family $(M_i)_{i \in I}$ of modules, then E is injective for $\bigoplus_{i \in I} M_i$.*

So by the observation that E is injective if and only if it is injective for every free module and using the fact that a free module is the direct sum of copies of R we get the Baer criterion for modules, i.e. E is injective if and only if E is injective for R.

Modifying these arguments we get the complex version of this criterion.

Theorem 1.4.11. *A complex E is injective if and only if it is injective for each $S^n(\overline{R})$.*

Note that subcomplexes of $\bar{R}$ are of the form $\cdots \to 0 \to I \hookrightarrow J \to 0 \to \cdots$ were I, J are left ideals of R.

1.5 Exercises

1. For a family $(C^i)_{i \in I}$ of complexes, find necessary and sufficient conditions in order that
$$\prod_{i \in I} C^i = \bigoplus_{i \in I} C^i$$
2. We say that a complex C is *cyclic* if there is an $x \in C$ that generates C (and then we say x is a *generator* of C).
   a) Find all cyclic complexes (where R is any ring)
   b) Let $R = \mathbb{Z}$ and let C and D be cyclic complexes. Find necessary and sufficient conditions in order that $C \oplus D$ be cyclic.
   c) Find all rings R for which it holds that every subcomplex of a cyclic complex (over R) is cyclic.
3. Show that every projective complex $P \neq 0$ has a subcomplex that is not projective.
4. Show that the ring R is left Noetherian if and only if every direct sum of injective complexes is injective.
5. If a complex C is the direct sum of its subcomplexes S and T, argue that for each $n \in \mathbb{Z}$, C_n is the direct sum of S_n and T_n. Find a counterexample to the converse of this claim.

6. If $S \subset C$ is a subcomplex of the complex C, we say that S is an *essential subcomplex* of C (or C is an *essential extension* of S) if $S \cap T \neq 0$ for every subcomplex $T \subset C$, $T \neq 0$. In this case we write $S'C$. Then show that C is an essential extension of S if and only if for every $x \in C$, $x \neq 0$ there is an $r \in R$ such either $rx \neq 0$ and $rx \in S$ or such that $rdx \neq 0$ and $rdx \in S$ where d is the differential.
7. Argue that for any complex C there is an injective complex E which is an essential extension of C (such an E is called an *injective envelope* of C).
8. If M is a module, find injective envelopes of $\bar{M}$ and of $\underline{M}$.
9. If C is a complex, show that $Z(C) \subset' C$ if and only if C has no subcomplexes isomorphic to $S^n(\bar{M})$ for $n \in \mathbb{Z}$, and $M \neq 0$ a module.
10. Call a complex C a *simple complex* if $C \neq 0$ and if C and 0 are the only subcomplexes of C. Find all simple complexes.
11. For a complex C, prove that the following are equivalent:
    a) every subcomplex $S \subset C$ is a direct summand of C
    b) C is the direct sum of simple subcomplexes of C
    c) $d = 0$ and each $C_n (n \in \mathbb{Z})$ is a direct sum of simple modules.
12. Show that if P is a projective complex then P is exact and each P_n is a projective module.
13. If k is a field, prove that every $C \in C(k\text{-Mod})$ is the direct sum of complexes of the form $S^k(\bar{k})$ and $S^j(\underline{k})$ (for $i, j \in \mathbb{Z}$).
14. For a ring R, argue that the following are equivalent:
    a) a module M is injective if and only if it is projective
    b) a complex C, is injective if and only if it is projective.
15. For a ring R, argue that the following are equivalent:
    a) every projective module is free
    b) every projective complex is free
16. If I is an exact complex and is such that for some n_0, $I_n = 0$ if $n > n_0$, prove that I is injective if and only if each I_k is an injective module.
17. Let $R = \mathbb{Z}/(4)$ and let $C = \cdots \to \mathbb{Z}/(4) \xrightarrow{2} \mathbb{Z}/(4) \xrightarrow{2} \mathbb{Z}/(4) \to \cdots$. Show that C is exact, has all its terms projective and all its terms injective, but that C is neither projective nor injective
18. Let $P \in C(\mathbb{Z}\text{-Mod})$ have all its terms finitely generated and projective (so free). Prove that P is the direct sum of complexes of some suspension of $\cdots \to 0 \to (n) \hookrightarrow \mathbb{Z} \to 0 \to \cdots$ with $n \in \mathbb{Z}$ and this $\mathbb{Z}$ in the 0^{th} place.
19. Give an example of a homology isomorphism $f : C \to D$ that is not an isomorphism in $C(R\text{-Mod})$ (cf. Definition 1.1.6).

Chapter 2
Short Exact Sequences of Complexes

For a ring R we will let $\mathcal{C}(R\text{-Mod})$ denote that category of complexes of left R-modules. So $C \in \mathcal{C}(R\text{-Mod})$ will mean that C is such a complex.

Given complexes C and D of left R-modules for some ring R, we introduce the groups $\text{Ext}^n(C, D)$ for $n \geq 0$. We show that the elements of $\text{Ext}^1(C, D)$ can be put in a bijective correspondence with the equivalence classes of short exact sequences $0 \to D \to U \to C \to 0$ of complexes.

We introduce the short exact sequences of complexes associated with the mapping cones of morphisms $f : C \to D$ in $\mathcal{C}(R\text{-Mod})$ and consider some of the properties of these sequences. We also consider the behavior of the homology groups associated with a short exact sequence of complexes.

2.1 The Groups $\text{Ext}^n(C, D)$

If $C, D \in \mathcal{C}(R\text{-Mod})$, we know there are exact sequence $P \to C \to 0$ and $0 \to D \to E$ in $\mathcal{C}(R\text{-Mod})$ where P is a projective complex and E is an injective complex. So we have the beginning of a projective resolution of C and of an injective resolution of D. Since we use subscripts to denote terms of a complex, we will use superscripts to distinguish the terms of these resolutions.

Definition 2.1.1. By a *projective resolution* of $C \in \mathcal{C}(R\text{-Mod})$, we mean an exact sequence of complexes

$$\cdots \to P^{-2} \to P^{-1} \to P^0 \to C \to 0$$

in $\mathcal{C}(R\text{-Mod})$ where each $P^{-n}, n \geq 0$, is a projective complex. By an *injective resolution* of $D \in \mathcal{C}(R\text{-Mod})$, we mean an exact sequence

$$0 \to D \to E^0 \to E^1 \to \cdots$$

of complexes in $\mathcal{C}(R\text{-Mod})$ where each E^n, $n \geq 0$, is an injective complex.

An exact sequence in $\mathcal{C}(R\text{-Mod})$ of the form

$$0 \to S \to P^{-(n-1)} \to \cdots \to P^0 \to C \to 0$$

with $P^0, P^{-1}, \ldots, P^{-(n-1)}$ projective complexes will be called a *partial projective resolution* of C of length n.

A *partial injective resolution* of $D \in C(R\text{-Mod})$ will be defined in a similar manner.

Example 2.1.2. If $M \in R$-Mod and if $\cdots \to P_1 \to P_0 \to M \to 0$ is a projective resolution of M in R-Mod, then using definition 1.2.1 we see that

$$\cdots \to \bar{P}_1 \to \bar{P}_0 \to \bar{M} \to 0$$

is a projective resolution of $\bar{M}$ in $C(R\text{-Mod})$.

Similarly for we can construct an injective resolution of $\bar{N} \in C(R\text{-Mod})$ from an injective resolution of N in R-Mod.

Once we have the notions of projective and injective resolutions in $C(R\text{-Mod})$, we can define the groups $\mathrm{Ext}^n(C, D)$ (or more precisely $\mathrm{Ext}^n_{C(R\text{-Mod})}(C, D)$ for $C, D \in C(R\text{-Mod})$).

So if

$$\cdots \to P^{-1} \to P^0 \to C \to 0$$

is a projective resolution of $C \in C(R\text{-Mod})$, then $\mathrm{Ext}^n(C, D)$ is defined to be the n^{th} *homology group* of the complex

$$0 \to \mathrm{Hom}(P^0, D) \to \mathrm{Hom}(P^1, D) \to \cdots$$

of Abelian groups.

The usual arguments give that these groups are well-defined. They can also be computed as the homology groups of

$$0 \to \mathrm{Hom}(C, E^0) \to \mathrm{Hom}(C, E^1) \to \cdots$$

where

$$0 \to D \to E^0 \to E^1 \to \cdots$$

is an injective resolution of D.

If $0 \to S \to P^{-(n-1)} \to \cdots \to P^{-1} \to P^0 \to C \to 0$ is a partial projective resolution of C then $\mathrm{Ext}^n(C, D)$ can be computed as the cokernel of

$$\mathrm{Hom}(P^{-(n-1)}, D) \to \mathrm{Hom}(S, D)$$

Using the complex version of the Horseshoe lemma and the fact that $\mathrm{Ext}^0(C, D) \cong \mathrm{Hom}(C, D)$, we get long exact sequences associated with short exact sequences $0 \to C' \to C \to C" \to 0$ and $0 \to D' \to D \to D'' \to 0$ of complexes. These are:

$$0 \to \mathrm{Hom}(C'', D) \to \mathrm{Hom}(C, D) \to \mathrm{Hom}(C', D) \to \mathrm{Ext}^1(C'', D) \to \cdots$$

and

$$0 \to \mathrm{Hom}(C, D') \to \mathrm{Hom}(C, D) \to \mathrm{Hom}(C, D'') \to \mathrm{Ext}^1(C, D') \to \cdots$$

Given a projective resolution

$$\cdots \to P^{-1} \to P^0 \to C \to 0$$

and a $k \in \mathbb{Z}$, we see that

$$\cdots \to S^k(P^{-1}) \to S^k(P^0) \to S^k(C) \to 0$$

is a projective resolution of $S^k(C)$.

For $D \in C(R\text{-Mod})$ we have $\mathrm{Hom}(S^k(P^{-n}), D) \cong \mathrm{Hom}(P^{-n}, S^{-k}(D))$. From this isomorphism it follows that

$$\mathrm{Ext}^n(S^k(C), D) \cong \mathrm{Ext}^n(C, S^{-k}(D))$$

for all $n \geq 0$.

Proposition 2.1.3. *If $M, N \in R$-Mod and $C \in C(R$-Mod), then* $\mathrm{Ext}^n(\bar{M}, C) \cong \mathrm{Ext}^n(M, C_1)$ *and* $\mathrm{Ext}^n(C, \bar{N}) \cong \mathrm{Ext}^n(C_{-1}, N)$ *for $n \in \mathbb{Z}$.*

Proof. Let $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$ be a projective resolution of M in R-Mod. Then $\cdots \to \bar{P}_2 \to \bar{P}_1 \to \bar{P}_0 \to \bar{M} \to 0$ is a projective resolution of $\bar{M}$ in $C(R\text{-Mod})$. We compute the $\mathrm{Ext}^n(\bar{M}, C)$ by considering the complex

$$0 \to \mathrm{Hom}(\bar{P}_0, C) \to \mathrm{Hom}(\bar{P}_1, C) \to \cdots$$

and computing homology. But for each k, $\mathrm{Hom}(\bar{P}_k, C) \cong \mathrm{Hom}(P_k, C_1)$. So this complex is isomorphic to the complex

$$0 \to \mathrm{Hom}(P_0, C_1) \to \mathrm{Hom}(P_1, C_1) \to \cdots$$

and the homology groups of this complex are the groups $\mathrm{Ext}^n(M, C_1)$. The second claim is proved in a similar manner. □

Proposition 2.1.4. *If $M, N \in R$-Mod and $C \in C(R$-Mod), then for $n, k \in \mathbb{Z}$*

$$\mathrm{Ext}^n(S^k(\bar{M}), C) \cong \mathrm{Ext}^n(M, C_{k+1})$$

and

$$\mathrm{Ext}^n(C, S^k(\bar{N})) \cong \mathrm{Ext}^n(C_{k-1}, N).$$

Proof. We have $\mathrm{Ext}^n(S^k(\bar{M}), C) \cong \mathrm{Ext}^n(\bar{M}, S^{-k}(C)) \cong \mathrm{Ext}^n(M, S^{-k}(C)_1) = \mathrm{Ext}^n(M, C_{k+1})$. The second isomorphism is proved in a similar manner. □

It is more complicated to compute the groups $\mathrm{Ext}^n(\underline{M}, C)$ and $\mathrm{Ext}^n(C, \underline{N})$. However, with $M = R$ we get the next result.

Proposition 2.1.5. *For $C \in \mathcal{C}(R\text{-Mod})$, we have*

$$\operatorname{Ext}^1(\underline{R}, C) \cong H_{-1}(C).$$

Proof. The complex $\bar{R}$ is projective (in fact it is free, cf. Example 1.3.4). The complex $\underline{R}$ is a subcomplex of $\bar{R}$ and $\bar{R}/\underline{R} \cong S(\underline{R})$. So we have the partial projective resolution $0 \to \underline{R} \to \bar{R} \to S(\underline{R}) \to 0$. Applying S^{-1} we get the partial projective resolution $0 \to S^{-1}(\underline{R}) \to S^{-1}(\bar{R}) \to \underline{R} \to 0$ of $\underline{R}$. By section 1.2, we know that $\operatorname{Hom}(S^{-1}(\underline{R}), C) \cong \operatorname{Hom}(\underline{R}, S(C)) \cong Z_{-1}(C)$. And also $\operatorname{Hom}(S^{-1}(\bar{R}), C) \cong \operatorname{Hom}(\bar{R}, S(C)) \cong C_0$.

With these isomorphisms, we have that

$$\operatorname{Hom}(S^{-1}(\bar{R}), C) \to \operatorname{Hom}(S^{-1}(\underline{R}), C)$$

corresponds to a map $C_0 \to Z_{-1}(C)$. But from the definition of the isomorphisms we see that $C_0 \to Z_{-1}(C)$ agrees with d_0 and so its cokernel is $H_{-1}(C)$. □

Using suspensions the next result is immediate.

Proposition 2.1.6. *For $k \in \mathbb{Z}$, and $C \in \mathcal{C}(R\text{-Mod})$*

$$\operatorname{Ext}^1(S^k(\underline{R}), C) \cong H_{k-1}(C).$$

Then we also get:

Corollary 2.1.7. *If $C \in \mathcal{C}(R\text{-Mod})$, then C is exact if and only if $\operatorname{Ext}^1(S^k(\underline{R}), C) = 0$ for all $k \in Z$.*

We will later need the next result.

Proposition 2.1.8. *If C is a finitely presented complex in $\mathcal{C}(R\text{-Mod})$, then for any directed system $(D^i : (f_{ji}))$ of complexes we have*

$$\operatorname{Ext}^n(C, \varinjlim D^i) \cong \varinjlim \operatorname{Ext}^n(C, D^i)$$

for all $n \geq 0$.

Proof. The proof is the same as that for modules. □

2.2 The Group $\operatorname{Ext}^1(C, D)$

In the first part of this section, we recall that if $M, N \in R$-Mod then the $\xi \in \operatorname{Ext}^1(M, N)$ can be put in bijective correspondence with the equivalence classes of short exact sequences $0 \to N \to U \to M \to 0$ in R-Mod. We will note that the same arguments can be applied to get such a correspondence between the $\xi \in \mathrm{Ext}^1(C, D)$ (for $C, D \in C(R\text{-Mod})$) and the equivalence classes of short exact sequences $0 \to D \to U \to C \to 0$ in $C(R\text{-Mod})$.

But first, we recall the following.

Definition 2.2.1. If we have a diagram

$$\begin{array}{ccc} S & \xrightarrow{f_1} & M_1 \\ {\scriptstyle p_2}\downarrow & & \\ M_2 & & \end{array}$$

of left R-modules, we can form the *pushout diagram*

$$\begin{array}{ccc} S & \xrightarrow{f_1} & M_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ M_2 & \xrightarrow{g_2} & P \end{array}$$

where $P = (M_1 \oplus M_2)/T$ with

$$T = \{(f_1(x), -f_2(x)) \mid x \in S\}$$

and where $g_1(x_1) = (x_1, 0) + T$ and $g_2(x_2) = (0, x_2) + T$ for $x_1 \in M_1$ and $x_2 \in M_2$.

This diagram can be expanded to a commutative diagram with exact rows:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{Ker}(f_1) & \hookrightarrow & S & \xrightarrow{f_1} & M_1 & \longrightarrow & \mathrm{Coker}(f_1) & \longrightarrow & 0 \\ & & & & {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Ker}(g_2) & \hookrightarrow & M_2 & \xrightarrow{g_2} & P & \longrightarrow & \mathrm{Coker}(g_2) & \longrightarrow & 0 \end{array}$$

It is easy to check that $\mathrm{Coker}(f_1) \to \mathrm{Coker}(g_2)$ is always an isomorphism and that if f_1 is injective, so is g_2.

Given any commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{f_1} & M_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1'} \\ M_2 & \xrightarrow{g_2'} & P' \end{array}$$

there is a unique linear map $h : P \to P'$ such that $g_1' = h \circ g_1$ and $g_2' = h \circ g_2$.

If furthermore this diagram is such that $\mathrm{Coker}(f_1) \to \mathrm{Coker}(g_2')$ is an isomorphism and such that f_1 and g_2' are injections, then we get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & M_1 & \longrightarrow & \mathrm{Coker}(f_1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_2 & \xrightarrow{g_2} & P & \longrightarrow & \mathrm{Coker}(g_2) & \longrightarrow & 0 \\ & & \| & & {\scriptstyle h}\downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_2 & \xrightarrow{g_2'} & P' & \longrightarrow & \mathrm{Coker}(g_2') & \longrightarrow & 0 \end{array}$$

with exact rows. Since we assumed $\mathrm{Coker}(f_1) \to \mathrm{Coker}(g_2')$ is an isomorphism, and since $\mathrm{Coker}(f_1) \to \mathrm{Coker}(g_2)$ is an isomorphism, we get that $\mathrm{Coker}(g_2) \to \mathrm{Coker}(g_2')$ is an isomorphism. This then implies that h is an isomorphism. So we also say that

$$\begin{array}{ccc} S & \xrightarrow{f_1} & M_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1'} \\ M_2 & \xrightarrow{g_2'} & P' \end{array}$$

is a pushout diagram.

So now with a change in notation we see that we have proved the next result.

Proposition 2.2.2. *Any diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & U & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & & & & & \\ & & S' & & & & & & \end{array}$$

of left R-modules with an exact row can be completed to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & U & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & S' & \longrightarrow & U' & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

with exact rows. Furthermore, given any such commutative diagram with exact rows,

$$\begin{array}{ccc} S & \longrightarrow & U \\ \downarrow & & \downarrow \\ S' & \longrightarrow & U' \end{array}$$

is a pushout diagram.

Definition 2.2.3. If $M, N \in R$-Mod, we say that two short exact sequences

$$\xi : 0 \to N \to U \to M \to 0$$

and

$$\xi' : 0 \to N \to U' \to M \to 0$$

are *equivalent* if there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & U & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & U' & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

We note that $U \to U'$ is then an isomorphism (and so we have an equivalence relation).

Definition 2.2.4. For $M, N \in R$-Mod let $\mathcal{E}xt(M, N)$ denote the set of all equivalence classes of short exact sequences $0 \to N \to U \to M \to 0$.

Theorem 2.2.5. *For $M, N \in R$-Mod, there is a bijection* $\mathrm{Ext}^1(M, N) \to \mathcal{E}xt(M, N)$.

Proof. Let $0 \to S \hookrightarrow P \to M \to 0$ be a partial projective resolution of M. This gives an exact sequence

$$\mathrm{Hom}(P, N) \to \mathrm{Hom}(S, N) \to \mathrm{Ext}^1(M, N) \to 0$$

We define a function $\mathrm{Hom}(S, N) \to \mathcal{E}xt(M, N)$ as follows:
If $f \in \mathrm{Hom}(S, N)$, we use a pushout to form a commutative diagram

$$\begin{array}{cccccccccc}
 & 0 & \longrightarrow & S & \hookrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\
 & & & {\scriptstyle f}\downarrow & & \downarrow & & \| & & \\
\xi : & 0 & \longrightarrow & N & \longrightarrow & U & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

We map f to the equivalence class $[\xi] \in \mathrm{Ext}(M, N)$.

Suppose $f, f' \in \mathrm{Hom}(S, N)$ have the same image in $\mathcal{E}xt(M, N)$. Then we have a diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S & \hookrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\
 & & {\scriptstyle f'}\downarrow{\scriptstyle f} & & {\scriptstyle g'}\downarrow{\scriptstyle g} & & \| & & \\
0 & \longrightarrow & N & \xrightarrow{h} & U & \xrightarrow{k} & M & \longrightarrow & 0
\end{array}$$

Since $g - g'$ maps P to the image of N in U, there is a linear $t : P \to N$ such that $h \circ t = g - g'$. This means that f and f' are in the same coset of $\mathrm{Im}(\mathrm{Hom}(P, N) \to \mathrm{Hom}(S, N))$ is $\mathrm{Hom}(S, N)$.

But then $h \circ (t \mid S) = h \circ (f - f')$ and so $t \mid S = f - f'$.

Conversely, if

$$\begin{array}{cccccccccc}
 & 0 & \longrightarrow & S & \hookrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\
 & & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \| & & \\
\xi : & 0 & \longrightarrow & N & \xrightarrow{h} & U & \xrightarrow{k} & M & \longrightarrow & 0
\end{array}$$

is a commutative diagram with exact rows and if $f' \in \mathrm{Hom}(S, N)$ is such that $f - f' = t \mid S$ for a linear map $t : P \to N$, then

$$\begin{array}{rccccccccc}
 & 0 & \longrightarrow & S & \hookrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\
 & & & {\scriptstyle f'}\big\downarrow & & {\scriptstyle g+h\circ t}\big\downarrow & & \big\| & & \\
\xi : & 0 & \longrightarrow & N & \xrightarrow{h} & U & \xrightarrow{k} & M & \longrightarrow & 0
\end{array}$$

is commutative. Hence f and f' have the same image in $\mathcal{E}xt(M, N)$. This shows that we get an injection $\mathrm{Coker}(\mathrm{Hom}(S, N) \to \mathrm{Hom}(P, N)) \to \mathcal{E}xt(M, N)$.

To show that this map is a bijection, let $\xi : 0 \to N \overset{h}{\to} U \overset{k}{\to} M \to 0$ be exact. Since P is projective, we can complete the diagram

$$\begin{array}{ccccc}
P & \longrightarrow & M & \longrightarrow & 0 \\
{\scriptstyle g}\big\downarrow & & \big\| & & \\
U & \xrightarrow{k} & M & \longrightarrow & 0
\end{array}$$

to a commutative diagram. But then there is a linear map $f : S \to U$ such that $h \circ (g \mid S) = h \circ f$. So the image of f in $\mathcal{E}xt(M, N)$ is the equivalence class of ξ. Hence $\mathrm{Hom}(S, N) \to \mathcal{E}xt(M, N)$ is surjective.

So finally we get that we have a bijection

$$\mathrm{Coker}(\mathrm{Hom}(S, N) \to \mathrm{Hom}(P, N)) \to \mathcal{E}xt(M, N).$$

The isomorphism

$$\mathrm{Coker}(\mathrm{Hom}(S, N) \to \mathrm{Hom}(R, N)) \cong \mathrm{Ext}^1(M, N)$$

then gives us the desired bijection

$$\mathrm{Ext}^1(M, N) \to \mathcal{E}xt(M, N). \qquad \square$$

Remark 2.2.6. The reader should check that another partial projective resolution $0 \to S' \hookrightarrow P' \to M \to 0$ of M leads to the same bijection $\mathrm{Ext}^1(M, N) \to \mathcal{E}xt(M, N)$. Also, if $f = 0$: $S \to N$, then the image of $f = 0$ is the equivalence class of $0 \to M \to M \oplus N \to N \to 0$, i.e. the split exact sequence.

Now if $C, D \in C(R\text{-Mod})$ and if we define $\mathcal{E}xt(C, D)$ to be the set of equivalence classes of short exact sequences $0 \to D \to U \to C \to 0$ in $C(R\text{-Mod})$, then with the same type arguments we get the following result.

Theorem 2.2.7. *For $C, D \in C(R\text{-Mod})$, there is a bijection*

$$\mathrm{Ext}^1(C, D) \to \mathcal{E}xt(C, D)$$

2.3 The Snake Lemma for Complexes

Given a commutative diagram

$$\begin{array}{ccccccccc}
 & & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & {\scriptstyle s'}\downarrow & & {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s''} & & \\
0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & &
\end{array}$$

of left R-modules with exact rows, there is a connecting homomorphism $\mathrm{Ker}(s'') \to \mathrm{Coker}(s')$ such that

$$\mathrm{Ker}(s') \to \mathrm{Ker}(s) \to \mathrm{Ker}(s'') \to \mathrm{Coker}(s') \to \mathrm{Coker}(s) \to \mathrm{Coker}(s'')$$

is an exact sequence. See Proposition 1.2.13 of Volume I.

The connecting homomorphism is natural. This means that if

$$\begin{array}{ccccccccc}
 & & \bar{M}' & \longrightarrow & \bar{M} & \longrightarrow & \bar{M}' & \longrightarrow & 0 \\
 & & {\scriptstyle \bar{s}'}\downarrow & & {\scriptstyle \bar{s}}\downarrow & & {\scriptstyle \bar{s}''}\downarrow & & \\
0 & \longrightarrow & \bar{N}' & \longrightarrow & \bar{N} & \longrightarrow & \bar{N}'' & &
\end{array}$$

is another such diagram, then any morphism of the first diagram into the second gives a commutative diagram

$$\begin{array}{ccccccccccc}
\mathrm{Ker}(s') & \longrightarrow & \mathrm{Ker}(s) & \longrightarrow & \mathrm{Ker}(s'') & \longrightarrow & \mathrm{Coker}(s') & \longrightarrow & \mathrm{Coker}(s) & \longrightarrow & \mathrm{Coker}(s'') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Ker}(\bar{s}') & \longrightarrow & \mathrm{Ker}(\bar{s}) & \longrightarrow & \mathrm{Ker}(\bar{s}'') & \longrightarrow & \mathrm{Coker}(\bar{s}') & \longrightarrow & \mathrm{Coker}(\bar{s}) & \longrightarrow & \mathrm{Coker}(\bar{s}'')
\end{array}$$

Using this observation we get our next result.

Proposition 2.3.1. *(The Snake Lemma for complexes). If*

$$\begin{array}{ccccccccc}
 & & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \\
 & & {\scriptstyle s'}\downarrow & & {\scriptstyle s}\downarrow & & {\scriptstyle s''}\downarrow & & \\
0 & \longrightarrow & D' & \longrightarrow & D & \longrightarrow & D'' & &
\end{array}$$

is commutative diagram in $C(R\text{-Mod})$ *with exact rows, then there is a connecting homomorphism* $\operatorname{Ker}(s'') \to \operatorname{Coker}(s')$ *such that the sequence*

$$\operatorname{Ker}(s') \to \operatorname{Ker}(s) \to \operatorname{Ker}(s'') \to \operatorname{Coker}(s') \to \operatorname{Coker}(s) \to \operatorname{Coker}(s'')$$

is exact.

Proof. For each $n \in \mathbb{Z}$, we have the commutative diagram

$$\begin{array}{ccccccc} & & C'_n & \longrightarrow & C_n & \longrightarrow & C''_n & \longrightarrow & 0 \\ & & {\scriptstyle s'_n}\downarrow & & {\scriptstyle s_n}\downarrow & & {\scriptstyle s''}\downarrow & & \\ 0 & \longrightarrow & D'_n & \longrightarrow & D_n & \longrightarrow & D''_n & & \end{array}$$

in R-Mod. Furthermore, the differentiations give a map of this diagram into the diagram corresponding to $n-1$. So we see that the connecting homomorphisms $\operatorname{Ker}(s''_n) \to \operatorname{Coker}(s'_n)$ give a map $\operatorname{Ker}(s'') \to \operatorname{Coker}(s')$ of complexes. Once we know this, the rest of the claim follows. □

Note that if $C \in C(R\text{-Mod})$, then the differentiation d of C can be regarded as a morphism $d : C \to S(C)$. So if $0 \to C' \to C \to C'' \to 0$ is a short exact sequence in $C(R\text{-Mod})$, we get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \\ & & {\scriptstyle d'}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle d''}\downarrow & & \\ 0 & \longrightarrow & S(C') & \longrightarrow & S(C) & \longrightarrow & S(C'') & \longrightarrow & 0 \end{array}$$

By Proposition 2.3.1, this gives rise to an exact sequence

$$\begin{aligned} 0 \to Z(C') \to Z(C) \to Z(C'') &\to S(C')/B(S(C')) \\ &= S(C'/B(C')) \to S(C/B(C)) \to S(C''/B(C'')) \to 0. \end{aligned}$$

So applying S^{-1} to this sequence, we get that

$$C'/B(C') \to C/B(C) \to C''/B(C'') \to 0$$

is exact. Again using the differentials to induce maps, we get a commutative diagram

$$\begin{array}{ccccccccc} & & C'/B(C') & \longrightarrow & C/B(C) & \longrightarrow & C''/B(C'') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S(Z(C')) & \longrightarrow & S(Z(C)) & \longrightarrow & S(Z(C'')) & & \end{array}$$

with exact rows. Applying the snake lemma to this diagram we get the exact sequence

$$H(C') \to H(C) \to H(C'') \to S(H(C')) \to S(H(C)) \to S(H(C'')).$$

Since S is an automorphism of $C(R\text{-Mod})$, we see that we also get the exact sequence

$$\begin{aligned} \cdots \to S^{-1}H(C')) \to S^{-1}H(C) \to S^{-1}H(C'') \to H(C') \\ \to H(C) \to H(C'') \to S(H(C')) \to S(H(C)) \to S(H(C'')) \to \cdots \end{aligned}$$

This exact sequence gives us the long exact sequence

$$\cdots \to H_{n+1}(C'') \to H_n(C') \to H_n(C) \to H_n(C'') \to H_{n-1}(C') \to \cdots .$$

Proposition 2.3.2. *If* $0 \to C' \to C \to C'' \to 0$ *is a short exact sequence in* $C(R\text{-Mod})$*, then if any two of* C'*,* C *and* C'' *are exact then so is the third.*

Proof. For a complex C, C exact means $H(C) = 0$. Clearly if $H(C) = 0$, then $H(S(C)) = 0$, and $H(S^{-1}(C)) = 0$. So the claim follows from the exactness of

$$\begin{aligned} H(S^{-1}(C')) \to H(S^{-1}(C) \to H(S^{-1}(C'')) \to H(C') \\ \to H(C) \to H(C'') \to H(S(C')) \to H(S(C)) \to H(S(C'')). \quad \square \end{aligned}$$

2.4 Mapping Cones

In the earlier sections of this chapter, we have considered short exact sequences of complexes. Mapping cones of morphisms in $C(R\text{-Mod})$ can be used to construct such short exact sequences.

Definition 2.4.1. Let $C, D \in C(R\text{-Mod})$ and let $f : C \to D$ be a morphism. Let $C(f)$ (called the *cone* of f) be the complex such that for each $n \in \mathbb{Z}$, $C(f)_n = D_n \oplus C_{n-1}$ and where

$$d_n^{C(f)}(y, x) = (d_n^D(y) + f_{n-1}(x), -d_{n-1}^C(x)).$$

We should check that we do have a complex. Using the simplified notation

$$d(y,x) = (d(y) + f(x), -d(x)),$$

we see that

$$d(d(y,x)) = (d^2(y) + d(f(x)) - f(d(x)), d^2(x)) = (0,0)$$

since $d \circ f = f \circ d$.

It is easy to check that we have morphisms $Y \to C(f)$ defined by $y \mapsto (y,0)$ and $C(f) \to S(C)$ defined by $(y,x) \mapsto x$. And then we see that we get a short exact sequence

$$0 \to Y \to C(f) \to S(C) \to 0$$

in $C(R\text{-Mod})$.

If we apply Proposition 2.3.1 to this short exact sequence, we get a morphism $H(S(C)) \to SH(Y) = H(S(Y))$.

Proposition 2.4.2. *A morphism* $f : C \to D$ *in* $C(R\text{-Mod})$ *is a homology isomorphism (see Definition* 1.1.6*) if and only if* $C(f)$ *is an exact complex.*

Proof. The short exact sequence $0 \to D \to C(f) \to S(C) \to 0$ gives rise to the exact sequence of homology groups

$$\cdots \to H(D) \to H(C(f)) \to H(S(C)) \\ \to H(S(D)) \to H(S(C(f))) \to H(S^2(C)) \to \cdots$$

Noting that f is a homology isomorphism if and only if $S(f)$ is a homology isomorphism the result follows. □

2.5 Exercises

1. a) Let $0 \to C' \to C \to C'' \to 0$ be a short exact sequence in $C(R\text{-Mod})$. Prove that $0 \to Z(C') \to Z(C) \to Z(C'')$ is also an exact sequence.
   b) Give an example where $Z(C) \to Z(C'') \to 0$ is not exact.
   c) Argue that if C' is exact, then $Z(C) \to Z(C'') \to 0$ is exact.
2. Again let $0 \to C' \to C \to C'' \to 0$ be an exact sequence in $C(R\text{-Mod})$. Argue that $0 \to B(C') \to B(C)$ and $B(C) \to B(C'') \to 0$ are exact, but in general $B(C') \to B(C) \to B(C'')$ is not exact.
3. Given $M, N \in R\text{-Mod}$, compute $\operatorname{Ext}^n(\bar{M}, \bar{N})$ and $\operatorname{Ext}^n(\underline{M}, \underline{N})$.

4. Define the *projective dimension* of $C \in C(R\text{-Mod})$ (with the notation $\operatorname{proj.dim} C$). Then prove that for $n \geq 0$, $\operatorname{proj.dim} C \leq n$ if and only if $\operatorname{Ext}^{n+1}(C, D) = 0$ for all $D \in C(R\text{-Mod})$.
5. If $\operatorname{proj.dim} C \leq n < +\infty$ for $C \in C(R\text{-Mod})$, argue that C is exact and that each term C_k of C has projective dimension at most n.
6. If $R \neq 0$ is a ring, show that there is always some $C \in C(R\text{-Mod})$ with $\operatorname{proj.dim} C = +\infty$.
7. Let $\operatorname{l.gl.dim} R \leq n < +\infty$. Argue that if $E \in C(R\text{-Mod})$ is exact, then $\operatorname{proj.dim} E \leq n$.
8. Prove 4.–7. above for *injective dimension* of complexes.
9. If $M, N \in R$-Mod, show that $\operatorname{Ext}^n(\bar{M}, \bar{N}) \cong \operatorname{Ext}^n(M, N)$ for all n. Then show also that $\operatorname{Ext}^n(\bar{M}, S(\bar{N})) \cong \operatorname{Ext}^n(M, N)$ for all n and that $\operatorname{Ext}^n(\bar{M}, S^k(\bar{N})) = 0$ if $k \neq 0, 1$.
10. If $C \in C(R\text{-Mod})$, a morphism $\underline{R} \to C$ is uniquely determined by some $x \in Z_0(C)$. Given such a morphism, compute the pushout of the diagram

$$\begin{array}{ccc} \underline{R} & \hookrightarrow & \overline{R} \\ \downarrow & & \\ C & & \end{array}$$

Argue that the pushout is a complex of the form

$$\cdots \to C_3 \xrightarrow{d_3} C_2 \to C_1 \oplus R \to C_0 \xrightarrow{d_0} C_{-1} \xrightarrow{d_{-1}} C_{-2} \to \cdots$$

where $C_2 \to C_1 \oplus R$ is the map $x \mapsto (d_2(x), 0)$ and where $C_1 \oplus R \to C_0$ is the map $(y, r) \mapsto d_1(y) + rx$. Then compute the homology modules of the pushout complex.
11. Let $S \subset C$ be a subcomplex of $C \in C(R\text{-Mod})$. Show that the following are equivalent:
    1) $0 \to H(S) \to H(C)$ is exact
    2) $H(C) \to H(C/S) \to 0$ is exact
    3) $B(S) = S \cap B(C)$
12. Let $C \in C(R\text{-Mod})$. Show that $Z(C) = 0$ if and only if $C = 0$.

Chapter 3
The Category $K(R\text{-Mod})$

3.1 Homotopies

Let R be a ring. We begin by defining the category of graded left R-modules.

Definition 3.1.1. By Gr(R-Mod), we mean the category whose objects are families $(M_n)_{n\in\mathbb{Z}}$ of left R-modules indexed by the set $\mathbb{Z}$ of integers. These objects will be called *graded* left R-modules. Given two such objects $M = (M_n)_{n\in\mathbb{Z}}$ and $N = (N_n)_{n\in\mathbb{Z}}$, by a morphism $f : M \to N$ of degree $p \in \mathbb{Z}$ we mean a family $(f_n)_{n\in Z}$ where for each n, $f_n : M_n \to N_{n+p}$ is a linear map. If $P = (P_n)_{n\in\mathbb{Z}}$ is another object and $g = (g_n)_{n\in Z} : N \to P$ is a morphism of degree q, then $g \circ f : M \to P$ is defined to be the family $(g_{n+p} \circ f_n)_{n\in\mathbb{Z}}$ with $g_{n+p} \circ f_n : M_n \to P_{n+p+q}$. So $g \circ f$ has degree $p + q$. So letting $\deg(f) = p$ mean f has degree p, we get the formula $\deg(g \circ f) = \deg(g) + \deg(f)$. So then for $M, N \in$ Gr(R-Mod), $\mathrm{Hom}_{\mathrm{Gr}(R\text{-Mod})}(M, N)$ (or simply $\mathrm{Hom}(M, N)$) is the set of morphisms $f : M \to N$ of any possible degree. We let $\mathrm{Hom}(M, N)_p \subset \mathrm{Hom}(M, N)$ denote the set of morphisms $f : M \to N$ with $\deg(f) = p$. We have that $f + g : M \to N$ is defined for $f, g \in \mathrm{Hom}(M, N)$ if $\deg(f) = \deg(g)$. So in fact $\mathrm{Hom}(M, N)$ is an object of Gr($\mathbb{Z}$-Mod).

If $f, f_1, f_2 \in \mathrm{Hom}(M, N)$ and $g, g_1, g_2 \in \mathrm{Hom}(N, P)$, we get the equalities

$$(g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f$$
$$g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2$$

when $\deg(g_1) = \deg(g_2)$ and $\deg(f_1) = \deg(f_2)$.

Using this terminology, we see that a complex $C \in C(R\text{-Mod})$ can be considered as an object of Gr(R-Mod) with a morphism $d = d^C : C \to C$ of degree -1 such that $d \circ d = d^C \circ d^C = 0$. So then a morphism $f : C \to D$ in $C(R\text{-Mod})$ is a morphism in Gr(R-Mod) such that $f \circ d^C = d^D \circ f$ (or $f \circ d = d \circ f$).

In Gr(R-Mod), we will also define $S(M)$ for $M = (M_n)_{n\in\mathbb{Z}}$ by the formula $S(M) = (M_{n-1})_{n\in\mathbb{Z}}$. For $f : M \to N$ of degree p we let $S(f) : S(M) \to S(N)$ be the morphism of degree P where $S(f) = (f_{n-1})_{n\in\mathbb{Z}}$.

Definition 3.1.2. If $C, D \in (R\text{-Mod})$ and if $f, g : C \to D$ are morphisms in $C(R\text{-Mod})$, we say f and g are *homotopic* if there is a morphism $s : C \to D$

in $\mathrm{Gr}(R\text{-Mod})$ of degree 1 so that $f - g = s \circ d^C + d^D \circ s$ (or succinctly, $f - g = s \circ d = d \circ s$). We write $f \cong g$ or $f \overset{s}{\equiv} g$ and say f is homotopic to g by s. We also say s is a *homotopy connecting* f and g.

Using the obvious notation, we give some properties of homotopies. These are $f \overset{0}{\cong} f$ (here $0 : C \to D$ is a morphism of degree 1), if $f \overset{s}{\cong} g$, then $g \overset{-s}{\cong} f$. If $f \overset{s}{\cong} g$ and $g \overset{t}{\cong} h$, then $f \overset{s+t}{\cong} g$. If $f \overset{s}{\cong} g$ and $f' \overset{s'}{\cong} g'$ then $f + f' \overset{s+s'}{\cong} g + g'$. If $f \overset{s}{\cong} g$ (say with $f, g : C \to D$) and if ($h : D \to E$ is a morphism is $C(R\text{-Mod})$, then $h \circ f \overset{h \circ s}{\cong} h \circ g$. Similarly $f \circ e \overset{s \circ e}{\cong} g \circ e$ for a morphism $e : B \to C$.

Proposition 3.1.3. *If $(C^i)_{i \in I}$ is a family of complexes in $C(R\text{-Mod})$ and if $f_i \overset{s_i}{\cong} g_i$ where $f_i, g_i \in \mathrm{Hom}(C^i, D)$ for some $D \in C(R\text{-Mod})$, then $f \overset{s}{\cong} g$ where $f, g : \bigoplus C^i \to D$ and $s : \bigoplus C^i \to D$ are defined by the formulas $f((x_i)_{i \in I}) = \sum_{i \in I} f(x_i)$, $g((x_i)_{i \in I}) = \sum_{i \in I} g_i(x_i)$ and $s((x_i)_{i \in I}) = \sum_{i \in I} s_i(x_i)$.*

Proof.

$$
\begin{aligned}
f((x_i)_{i \in I}) - g((x_i)_{i \in I}) &= \sum_{i \in I} f_i(x_i) - \sum_{i \in I} g_i(x_) \\
&= \sum_{i \in I} (f_i(x_i)) - g_i(x_i)) = \sum_{i \in I} (d^D \circ s_i + s_i \circ d_i^{C_i})(x_i) \\
&= d^D\Big(\sum_{i \in I} s_i(x_i)\Big) + S((d^{C^i}(x_i)_{i \in I}) \\
&= d^D \circ s((x_i)_{i \in I}) + s(d^{\bigoplus_{i \in I} C_i}((x_i)_{i \in I}))
\end{aligned}
$$

So we get

$$
d \circ s + s \circ d = f - g. \qquad \square
$$

Now given a family $(D^j)_{j \in I}$ of complexes of left R-modules and morphisms $f_j, g_j : C \to D^j$ for each $j \in I$ and homotopies $f_j \overset{s_j}{\cong} g_j$ for each $j \in J$, we get morphisms $f, g : C \to \prod_{j \in I} D^j$ and a homotopy s with $f \overset{s}{\cong} g$. These claims are easily verified.

3.2 The Category $K(R\text{-Mod})$

Given a ring R, and $C, D \in C(R\text{-Mod})$, we have an equivalence relation $f \cong g$ (f is homotopic to g) on the $f, g \in \mathrm{Hom}_{C(R\text{-Mod})}(C, D) = \mathrm{Hom}(C, D)$. We let $[f]$ denote the equivalence class of f. If $f_1, f_2 \in \mathrm{Hom}(C, D), g_1, g_2 \in \mathrm{Hom}(D, E)$

with $E \in C(R\text{-Mod})$, then $f_1 \cong f_2$ and $g_1 \cong g_2$ imply that $g_1 \circ f_1 \cong g_2 \circ f_2$. This means that for $f : C \to D$ and $g : D \to E$, we can define $[g] \circ [f]$ as $[g \circ f]$. Similarly we can define $[f] + [g]$ to be $[f + g]$ when $f, g \in \mathrm{Hom}(C, D)$. So we see that we get a category.

Definition 3.2.1. For a ring R, we let $K(R\text{-Mod})$ be the category whose objects are the $C \in C(R\text{-Mod})$ and whose morphisms $C \to D$ are the equivalence classes $[f]$ for $f \in \mathrm{Hom}_{C(R\text{-Mod})}(C, D)$.

To distinguish the sets $\mathrm{Hom}_{C(R\text{-Mod})}(C, D)$ and the sets $\mathrm{Hom}_{K(R\text{-Mod})}(C, D)$ we let $\mathrm{Hom}(C, D)$ mean $\mathrm{Hom}_{C(R\text{-Mod})}(C, D)$. Note that each $\mathrm{Hom}_{K(R\text{-Mod})}(C, D)$ is an abelian group.

In $K(R\text{-Mod})$, we have an identity $[\mathrm{id}_C]$ where id_C is in $C(R\text{-Mod})$). We will denote $[\mathrm{id}_C]$ simply as id_C.

So if $f : C \to D$ is an isomorphism in $C(R\text{-Mod})$, then $[f]$ is an isomorphism in $K(R\text{-Mod})$ with $[f]^{-1} = [f^{-1}]$.

Many properties of $C(R\text{-Mod})$ are lost when we consider $K(R\text{-Mod})$. For example, there are no natural notions of $\mathrm{Ker}([f])$ or $\mathrm{Coker}([f])$ for $[f] \in \mathrm{Hom}_{K(R\text{-Mod})}(C, D)$.

However, if $(C^i)_{i\in I}$ is a family of complexes in $C(R\text{-Mod})$, then we can form the complex $\bigoplus_{i\in I} C^i$. Then this complex with the canonical homomorphisms $e_j : C^j \to \bigoplus_{i\in I} C^i$ are an (external) *direct sum* in $C(R\text{-Mod})$. This means that given morphisms $f_j : C^j \to D$ for each $j \in I$, there is a unique morphism $f : \bigoplus_{i\in I} C^i \to D$ such that $f \circ e_j = f_j$ for each $j \in I$.

Using this notation we get the next result.

Proposition 3.2.2. *With the morphisms* $[e_j] : C^j \to \bigoplus_{i\in I} C^i$, $\bigoplus_{i\in I} C^i$ *is a direct sum in* $K(R\text{-Mod})$.

Proof. Given $D \in K(R\text{-Mod})$ and $[f_j] : C^j \to D$ for each $j \in I$, we have the morphisms $f_j : C^j \to D$ in $C(R\text{-Mod})$.

These give a morphism $f : \bigoplus_{i\in I} C^i \to D$ so that $f \circ e_j = f_j$ for each $j \in I$. So $[f] \circ [e_j] = [f_j]$ for each $j \in I$. To establish the uniqueness of f, suppose $g : \bigoplus_{i\in I} C^i \to D$ is such that $[g]\circ[e_j] = [f_j]$ for each j. This gives that $g \circ e_j \overset{s_j}{\cong} f_j$ for some homotopy s_j. Then by Proposition 3.1.3 we see that this gives that $g \cong f$, i.e. that $[g] = [f]$. □

In a similar manner, we get products in $K(R\text{-Mod})$. We let $p_j : \prod_{i\in I} C^i \to C^j$ for $j \in I$ be the canonical projections.

Proposition 3.2.3. *With the morphisms* $[p_j] : \prod_{i\in I} C^i \to C^j$, $\prod_{i\in I} C^i$ *is a product in* $K(R\text{-Mod})$.

We noted that if $C, D \in C(R\text{-Mod})$ are isomorphic in $C(R\text{-Mod})$ with $f : C \to D$ an isomorphism, then $[f] : C \to D$ is an isomorphism in $K(R\text{-Mod})$ with $[f]^{-1} = [f^{-1}]$. However, $[f] : C \to D$ can be an isomorphism in $K(R\text{-Mod})$ where $f : C \to D$ is not an isomorphism. The morphism $[f]$ will be an isomorphism in $K(R\text{-Mod})$ if and only if there is a $g : D \to C$ in $C(R\text{-Mod})$ such that $[g] \circ [f] = [\mathrm{id}_C]$ and $[f] \circ [g] = [\mathrm{id}_D]$, or equivalently if and only if $g \circ f \cong \mathrm{id}_C$ and $f \circ g \cong \mathrm{id}_D$. Such an f will be called a *homotopy isomorphism*.

We now characterize the $C \in C(R\text{-Mod})$ that are isomorphic to the complex in $K(R\text{-Mod})$.

Proposition 3.2.4. *A complex $C \in C(R\text{-Mod})$ is isomorphic to 0 in $K(R\text{-Mod})$ if and only if C is the direct sum of complexes of the form $S^k(\bar{M})$ where $M \in R\text{-Mod}$ and $k \in \mathbb{Z}$.*

Proof. We first argue that for $M \in R\text{-Mod}$, $\bar{M}$ is isomorphic to 0. Since $f = 0 : \bar{M} \to 0$ and $g = 0 : 0 \to \bar{M}$ are the only possible morphisms in $C(R\text{-Mod})$, this means we only need show that $id_{\bar{M}} \overset{s}{\cong} 0$ for some s. But if we let $s = (s_n)_{n \in \mathbb{Z}}$ be such that $s_0 = \mathrm{id}_M$ and such that $s_n = 0$ if $n \neq 0$ then we have $\mathrm{id}_{\bar{M}} \overset{s}{\cong} 0$. In a similar manner, we see any $S^k(\bar{M})$ is isomorphic to 0 in $K(R\text{-Mod})$. Using Proposition 3.2.2, we see that any direct sum of $S^k(\bar{M})$'s is also isomorphic to 0 in $K(R\text{-Mod})$

Conversely, suppose $C \in C(R\text{-Mod})$ is isomorphic to 0 in $K(R\text{-Mod})$. Then $\mathrm{id}_C \overset{s}{\equiv} 0$. So for each $n \in \mathbb{Z}$ we have

$$\mathrm{id}_{C_n} = s_{n-1} \circ d_n + d_{n+1} \circ s_n.$$

Restricting to $Z_n(C)$, this equality becomes

$$\mathrm{id}_{Z_n(C)} = d_{n+1} \circ (s_n \mid Z_n(C)).$$

This gives that $C_{n+1} \to Z_n(C)$ is surjective and has a section. So $Z_{n+1}(C)$ is a direct summand of C_{n+1} with a complementary submodule S_{n+1} such that $S_{n+1} \to Z_n(C)$ is an isomorphism. So $\cdots \to 0 \to S_{n+1} \overset{\cong}{\to} Z_n(C) \to 0$ is a subcomplex of C. Since C is the direct sum of these subcomplexes, we have established our claim. □

3.3 Split Short Exact Sequences

Given a short exact sequence $0 \to C' \overset{f}{\to} C \overset{g}{\to} C'' \to 0$ in $C(R\text{-Mod})$, we will be concerned with criteria that guarantee the sequence is split exact in $C(R\text{-Mod})$.

Just as for modules, we have the splitting criterion:

Proposition 3.3.1. *If*

$$0 \to C' \xrightarrow{f} C \xrightarrow{g} C'' \to 0$$

is a short sequence in $C(R\text{-Mod})$*, then the following are equivalent:*

a) f *admits a retraction* $r : C \to C'$ *(so* $r \circ f = \mathrm{id}_{C'}$*)*

b) g *admits a section* $t : C'' \to C$ *(so* $g \circ t = \mathrm{id}_{C''}$*).*

We note that if $0 \to C' \xrightarrow{t} C \xrightarrow{g} C'' \to 0$ is split exact with a retraction $r : C \to C'$, then for each $n \in \mathbb{Z}$,

$$0 \to C'_n \xrightarrow{f_n} C_n \xrightarrow{g_n} C''_n \to 0$$

is split exact with a retraction $r_n : C_n \to C'_n$.

It can happen that each $0 \to C'_n \to C_n \to C''_n \to 0$ is split exact without $0 \to C' \to C \to C'' \to 0$ being split exact. If we only know that each $0 \to C'_n \to C_n \to C''_n \to 0$ is split exact we say that $0 \to C' \to C \to C'' \to 0$ is split exact at the module level. If this is the case then each $0 \to C'_n \to C_n \to C''_n \to 0$ is isomorphic to the short exact sequence

$$0 \to C'_n \to C'_n \oplus C''_n \to C''_n \to 0.$$

Replacing C_n with $C'_n \oplus C''_n$ we see that the $d_n : C_n \to C_{n-1}$ (i.e. $d_n : C'_n \oplus C''_n \to C_{n-1} \oplus C''_{n-1}$) must be of the form

$$d_n(x', x'') = (d'_n(x') + f_n(x''), d''_n(x''))$$

where $f_n : C''_n \to C'_{n-1}$ is a linear map. Since $d_{n-1} \circ d_n = 0$ we see that we have

$$d_n(f_n(x'')) + f_{n-1}(d_n(x'')) = 0.$$

This means that we can use the f_n to create a morphism $g : S^{-1}(C'') \to C'$ where $g_n = f_{n+1}$. Then we check that $C = C(g)$. So, up to isomorphism, an exact sequence $0 \to C' \to C \to C'' \to 0$ that splits at the module level is the short exact sequence associated with a mapping cone. So then we ask what more is needed to give that the sequence is split exact.

Proposition 3.3.2. *If* $f : C \to D$ *is a morphism in* $C(R\text{-Mod})$*, then*

$$0 \to D \to C(f) \to S(C) \to 0$$

is split exact if and only if $f \cong 0$*.*

Proof. Assume
$$0 \to D \to C(f) \to S(C) \to 0$$
is split exact. Let $t : S(C) \to C(f)$ be a section. This means that for $x \in S(C)_n = C_{n-1}$, $t_n(x) = (u_n(x), x)$ where $u_n : C_{n-1} \to D_n$ is linear for each n.

Since $t : S(C) \to C(f)$ is a morphism in $C(R\text{-Mod})$, we now drop subscripts and let $x \in S(C)$ (i.e. $x \in S(C)_n$) for a unique $n \in \mathbb{Z}$). Then the fact that $t : S(C) \to C(f)$ is a morphism of complexes gives the equation

$$(-u(d^C(x)), -d^C(x)) = (d^D(u(x)) + f(x), -d^C(x))$$

and so that
$$f(x) = -d^D(u(x)) - u(d^C(x)).$$
But this gives that if $s = -u$, then we have $f \overset{s}{\cong} 0$.

These steps can be reversed and give that $f \cong 0$ imply

$$0 \to D \to C(f) \to S(0) \to 0$$

has a section. □

It turns out that it useful to have an even weaker condition that guarantees a mapping cone sequence splits.

Proposition 3.3.3. *If* $f : C \to D$ *is a morphism in* $C(R\text{-Mod})$ *then*

$$0 \to D \overset{f}{\to} C(f) \to S(C) \to 0$$

is split exact if and only if the morphism $[f]$ *in* $K(R\text{-Mod})$ *admits a retraction in* $K(R\text{-Mod})$.

Proof. We drop subscripts for complexes in our proof and also write compositions $g \circ f$ as products gf.

Let $r : C(f) \to D$ be such that $[r]$ is a retraction for $[u]$. This means that $\mathrm{id}_D \overset{t}{\cong} r \circ s$ for some homotopy t. This in turn means that for $y \in D$ we have $(dt + td)(y) = y - r(y, 0)$. Define $C(f) \to D$ by $(y, x) \mapsto y + t(f(x)) + r(0, x)$ for $(y, x) \in C(f)$. If this is a morphism of complexes, it gives a retraction of u in $C(R\text{-Mod})$. Since $d(y, x) = (dy + f(x), -dx)$, we need to show that

$$d(y + t(f(x)) + r(0, x)) = dy + f(x) - t(f(dx)) - r(0, dx).$$

Canceling dy and using the fact that $df = fd$ in the term $-t(f(dx))$, we see that we need to show that

$$d(t(f(x)) + t(d(f(x)) = f(x) - dr(0, x) - r(0, dx).$$

But $d(t(f(x)) + (d(f(x))) = (dt + td)(f(x)) = f(x) - r(f(x), 0)$. So now canceling $f(x)$, we see that we need

$$r(f(x), 0) = dr(0, x) + r(0, dx).$$

Since $dr = rd$ we have

$$\begin{aligned} dr(0, x) + r(0, dx) &= rd(0, x) + r(0, dx) \\ &= r(f(x), -dx) + r(0, dx) \\ &= r((f(x), 0) + (0, -dx)) + r(0, dx) \\ &= r(f(x), 0) \end{aligned}$$

as desired. □

We have a dual result.

Proposition 3.3.4. *If $f : C \to D$ is a morphism in $C(R\text{-Mod})$ then $0 \to D \to C(f) \xrightarrow{q} S(C) \to 0$ is split exact if and only if the morphism $[q]$ in $K(R\text{-Mod})$ admits a section in $K(R\text{-Mod})$.*

Proof. We argue that if we have a section up to homotopy then we have a section.

Let $s : S(C) \to C(f)$ be a section up to homotopy where $s(x) = (u(x), v(x))$. Let w be the associated homotopy. Then $-(dw + wd)(x) = x - v(x)$. We have

$$ds(x) = d(u(x), v(x)) = (du(x) + fv(x), -dv(x))$$

But s is a morphism and so $ds(x) = s(-dx) = (-ud(x), -vd(x))$. So we get $(du + ud)(x) = -fv(x)$ and $dv(x) = vd(x)$.

We now claim that $x \mapsto (fw(x) + u(x), x)$ is the desired section. To get that this function commutes with differentials we need that $dfw(x) + du(x) + f(x) = -fwd(x) - ud(x)$ or that $f((dw + wd)(x)) + (du + ud)(x) = -f(x)$. Since $(dw + wd)(x) = v(x) - x$ and since $(du + ud)(x) = -fv(x)$, we see that the equality holds. □

3.4 The Complexes $\mathcal{H}om(C, D)$

Given $C, D \in C(R\text{-Mod})$ we can regard C and D as objects of $\text{Gr}(R\text{-Mod})$. So we form the graded abelian group $\text{Hom}_{\text{Gr}(R\text{-Mod})}(C, D)$. We will show that this graded abelian group can be made into a complex. To simplify notation we will write $\mathcal{H}om(C, D)$ in place of $\text{Hom}_{\text{Gr}(R\text{-Mod})}(C, D)$.

Then instead of $d^{\mathcal{H}om(C,D)}$ for the differential, we will write d'.

Definition 3.4.1. For $C, D \in C(R\text{-Mod})$, we let $\mathcal{H}om(C, D)$ be the complex with differential d' where $d'(f) = f \circ d^C - (-1)^p d^D \circ f$ when $f \in \mathcal{H}om(C, D)_p$.

We need to check that d' is indeed a differentiation. We first note that $d'(f)$ is defined since $f \circ d^C$ and $d^D \circ f$ both have degrees $p - 1$.

We have

$$\begin{aligned} d'^2(f) = d'(d'(f)) &= d'(f \circ d^C - (-1)^p d^D \circ f) \\ &= (f \circ d^C - (-1)^p d^D \circ f) \circ d^C \\ &\quad - (-1)^{p-1} d^D \circ (f \circ d^C - (-1)^P d^D \circ f) \end{aligned}$$

Since $d^C \circ d^C = 0$ and $d^D \circ d^D = 0$ and since $-(-1)^p d^D \circ f \circ d^C - (-1)^{p-1} d^D \circ f \circ d^C = 0$, we see that $d'^2(f) = 0$.

Proposition 3.4.2. *If* $C, D \in C(R\text{-Mod})$*, then* $f \in \mathcal{H}om(C, D)_p$ *will be a cycle (i.e.* $f \in Z_p(\mathcal{H}om(C, D))$ *if and only if* $f \in \mathrm{Hom}_{C(R\text{-Mod})}(C, S^p(D))$.

Proof. Suppose $f \in Z_p(\mathcal{H}om(C, D))$. Since $\deg(f) = p$, we can regard f as a morphism $C \to S^p(D)$ of graded modules of degree 0. The equation $d'(f) = 0$ says

$$f \circ d^C - (-1)^p d^D \circ f = 0$$

i.e. that $f \circ d^C = (-1)^p d^D \circ f$. Since $(-1)^p d^D$ is the differentiation of $S^p(D)$ we see that $f : C \to S^p(D)$ is a morphism in $C(R\text{-Mod})$. Reversing the steps we get the converse. □

We now characterize the boundaries in $\mathcal{H}om(C, D)$.

Proposition 3.4.3. *For* $C, D \in C(R\text{-Mod})$ *and* $f \in \mathcal{H}om(C, D)_p$ *we have* f *is a boundary of* $\mathcal{H}om(C, D)$ *(i.e.* $f \in B_p(\mathcal{H}om(C, D))$ *if and only if* f *as a morphism* $C \to S^p(D)$ *in* $C(R\text{-Mod})$ *is homotopic to* 0.

Proof. Assume f is a boundary. Then f is a cycle, so by the preceding result $f : C \to S^p(D)$ is a morphism. If f is a boundary, then $d'(s) = f$ with $s \in \mathcal{H}om(C, D)_{p+1}$. But this means

$$\begin{aligned} f &= s \circ d^C - (-1)^{p+1} d^D \circ s \\ &= s \circ d^C + (-1)^p d^D \circ s \end{aligned}$$

Noting that s can be regarded as a map of graded modules $C \to S^P(D)$ of degree $+1$ we see that $f \cong 0$.

We get the converse by reversing the steps. □

Given complexes C and D of left R-modules, we form the complex $\mathcal{H}om(C, D)$. It is then natural to ask about the properties of this complex. We can now see when $\mathcal{H}om(C, D)$ is exact.

Corollary 3.4.4. *For $C, D \in C(R\text{-Mod})$, $\mathcal{H}om(C, D)$ is an exact complex if and only if for every $p \in \mathbb{Z}$ every morphism $f : C \to S^p(D)$ in $C(R\text{-Mod})$ is homotopic to* 0.

Proof. This result follows immediately from Propositions 3.4.2 and 3.4.3 above. □

Given $C, D \in C(R\text{-Mod})$ and $f \in \text{Hom}(C, D)$, we can construct the mapping cone $C(f)$ and the associated exact sequence $0 \to D \to C(f) \to S(C) \to 0$. By Theorem 2.2.7, this exact sequence can be regarded as an element of $\text{Ext}^1(S(C), D)$. If we know that $\text{Ext}^1(S(C), D) = 0$, then the sequence $0 \to D \to C(f) \to S(C) \to 0$ is split exact. By Proposition 3.3.2 this sequence is split exact if and only if f is homotopic to 0. So we get that if $\text{Ext}^1(S(C), D)) = 0$ then every morphism $f : C \to D$ is homotopic to 0. We would have the converse of this statement if we knew that every element of $\text{Ext}^1(S(C), D))$ thought of as a short exact sequence. $0 \to D \to U \to S(C) \to 0$ were equivalent to a sequence $0 \to D \to C(f) \to S(C) \to 0$ associated with a morphism $f : C \to D$. In Section 3.3 it was noted that if $0 \to D \to U \to S(C) \to 0$ splits at the module level then the sequence is equivalent to a mapping cone short exact sequence.

Finally we remark that for $C, D \in C(R\text{-Mod})$, the condition that every morphism $f : C \to D$ in $C(R\text{-Mod})$ is homotopic to 0 is equivalent to the condition that $\text{Hom}_{K(R\text{-Mod})}(C, D) = 0$.

3.5 The Koszul Complex

In this section, we let R be a commutative ring and $r \in R$. Let K be the complex $\cdots \to 0 \to R \xrightarrow{r} R \to 0 \to \cdots$ with the R's in the 1st and 0th positions. Then we have the following.

Lemma 3.5.1. *For $D \in C(R\text{-Mod}), \text{Hom}_{K(R\text{-Mod})}(K, D) = 0$ if and only if $0 \to H_0(D) \xrightarrow{r} H_0(D)$ and $H_1(D) \xrightarrow{r} H_1(D) \to 0$ are exact.*

Proof. The morphisms $f : K \to D$ correspond to a choice of $y \in D$ and $x \in D_0$ subject to the conditions that $dx = 0$ and $dy = rx$. So this means x is a cycle and rx is a boundary. A homotopy $s : K \to D$ is determined by a choice of $u \in D_2$ and $v \in D_1$.

We first assume $\text{Hom}_{K(R\text{-Mod})}(K, D) = 0$. So every $f : K \to D$ is homotopic to 0. We argue that

$$0 \to H_0(C) \xrightarrow{r} H_0(C)$$

is injective. If $x+B_0(C)$ (with $x \in Z_0(C)$) is in the kernel of $H_0(C) \xrightarrow{r} H_0(C)$, then rx is a boundary. So let $dy = rx$. Such an x and y give us a morphism $f : K \to D$. By hypothesis $f \overset{s}{\equiv} 0$. Let s correspond to $u \in D_2$ and $v \in D_1$. Then $f \overset{s}{\equiv} 0$ gives us the equations $dv = x$ and $du + rv = y$. In particular, we see that $x \in B_0(C)$ and so that $x + B_0(C) = 0$. So $0 \to H_0(C) \xrightarrow{r} H_0(C)$ is exact.

We now argue that $H_1(D) \xrightarrow{r} H_1(D) \to 0$ is exact. Let $y + B_1(D) \in H_1(D)$ (with $y \in Z_1(D)$). Then since $dy = 0$, with this $y \in D_1$ and $x = 0 \in D_0$ we get a morphism $f : K \to D$. Then $f \cong 0$ just says that for some $u \in D_2$, $v \in D$, $dv = x = 0$ and $du + rv = y$. So $v \in Z_1(D)$ and we get $r(v + B_1(D)) = y + B_1(D)$. So $H_1(D) \xrightarrow{r} H_1(D) \to 0$ is exact.

Conversely assume that $0 \to H_0(D) \xrightarrow{r} H_0(D)$ and $H_1(D) \xrightarrow{r} H_1(D) \to 0$ are exact. We want to argue that every morphism $f : K \to D$ is homotopic to 0. Let f correspond to $y \in D_1$, $x \in D_0$ with $dx = 0$ and $dy = rx$. Since $0 \to H_0(D) \xrightarrow{r} H_0(D)$ is injective and since $r(x + B_0(D)) = rx + B_0(D) = dy + B_0(D) = 0$, we get $x + B_0(D) = 0$. So $x = dv'$ for $v' \in D_1$. But the $drv' = rx$. So since $dy = rx$, we get $d(y - rv') = 0$, i.e.

$$y - rv' \in Z_1(D)$$

Then since $H_1(D) \xrightarrow{r} H_1(D) \to 0$ is exact, there is a $z + B_1(D) \in H_1(D)$ with $rz + B_1(D) = y - rv' + B_1(D)$. This means $y - rv' = rz + du$ for some $u \in D_2$ and so that $y = r(v' + z) + du$. If $v = v' + z$, then $dv = dv' + dz = x + 0$ and $y = rv + du$. So this gives that $f \cong 0$. □

3.6 Exercises

1. Let $D \in C(R\text{-Mod})$. Prove that $\mathcal{H}om(\bar{M}, D)$ is exact for all $M \in R\text{-Mod}$.
2. Find a necessary and sufficient condition on $C \in C(R\text{-Mod})$ such that $0 \to \mathcal{H}om(C, D') \to \mathcal{H}om(C, D) \to \mathcal{H}om(C, D'') \to 0$ is exact for all exact $0 \to D' \to D \to D'' \to 0$ in $C(R\text{-Mod})$.
3. For $M, N \in R\text{-Mod}$, explain why $\mathcal{H}om(\underline{M}, \underline{N}) \cong \underline{\mathrm{Hom}(M, N)}$.
4. Let $r \in R$ where R is a commutative ring and let $K \in C(R\text{-Mod})$ be as in Section 3.5. If r is nilpotent (i.e. $r^n = 0$ for some $n \geq 1$) and if $D \in C(R\text{-Mod})$, show that $\mathcal{H}om(K, D)$ is exact if and only if D is exact.
5. Let $f : C \to D$, $g : D \to E$ be morphisms in $C(R\text{-Mod})$. Define natural morphisms $C(f) \to C(g \circ f)$ and $C(g \circ f) \to C(g)$.

Chapter 4
Cotorsion Pairs and Triplets in $C(R\text{-Mod})$

In this chapter we give the basic results concerning cotorsion pairs of classes of complexes of left R-modules.

4.1 Cotorsion Pairs

Definition 4.1.1. If $\mathcal{A}$ is a class of complexes of left R-modules we let $\mathcal{A}^\perp$ consist of all $C \in C(R\text{-Mod})$ such that $\mathrm{Ext}^1(A, C) = 0$ for all $A \in \mathcal{A}$. We let ${}^\perp\mathcal{A}$ consist of all $B \in C(R\text{-Mod})$ such that $\mathrm{Ext}^1(B, A) = 0$ for all $A \in \mathcal{A}$.

Definition 4.1.2. A pair $(\mathcal{A}, \mathcal{B})$ of classes of objects of $C(R\text{-Mod})$ is said to be a *cotorsion pair* (or a *cotorsion theory*) for $C(R\text{-Mod})$ if $\mathcal{B}^\perp = \mathcal{B}$ and ${}^\perp\mathcal{B} = \mathcal{A}$.

These notions in $C(R\text{-Mod})$ are basically the same as those for R-Mod (see Chapter 7 of the Volume I). Some arguments and ideas there carry over to the category $C(R\text{-Mod})$ with little modification.

Definition 4.1.3. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $C(R\text{-Mod})$ is said to be *complete* (or to have *enough injectives and projectives*) if for every $C \in C(R\text{-Mod})$ there are exact sequences $0 \to B \to A \to C \to 0$ and $0 \to C \to B' \to A' \to 0$ in $C(R\text{-Mod})$ with $A, A' \in \mathcal{A}$ and $B, B' \in \mathcal{B}$.

If we only assume the existence of the sequences $0 \to B \to A \to C \to 0$ (or the sequences $0 \to C \to B' \to A' \to 0$) for every C, then the argument in Volume I (see Proposition 7.1.7) shows we also get the sequences $0 \to C \to B' \to A' \to 0$ (the sequences $0 \to B \to A \to C \to 0$).

The best result guaranteeing that a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $C(R\text{-Mod})$ was provided by Eklof–Trlifaj. We recall that result below.

We note that if $\mathcal{S}$ is any class of objects of $C(R\text{-Mod})$ and if $\mathcal{S}^\perp = \mathcal{B}$ and $\mathcal{A} = {}^\perp\mathcal{B}$, then $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair. We say it is the *cotorsion pair cogenerated* by $\mathcal{S}$. If there is a set $\mathcal{S}$ that cogenerates $(\mathcal{A}, \mathcal{B})$, then we say that $(\mathcal{A}, \mathcal{B})$ is *cogenerated* by a set.

The next result was originally proved for modules. A proof for modules is given in Volume I (see Theorem 7.4.1). The proof carries over directly to complexes.

Theorem 4.1.4 (Eklof, Trlifaj). *If a cotorsion pair* $(\mathcal{A}, \mathcal{B})$ *in* $C(R\text{-Mod})$ *is cogenerated by a set, then it is complete.*

The proof of this theorem allows us to give a description of $\mathcal{A}$ in terms of $\mathcal{S}$ when $(\mathcal{A}, \mathcal{B})$ is the cotorsion pair cogenerated by the set $\mathcal{S}$. To give this description we will need some terminology.

Some of this terminology is in a state of flux and so will not be precisely that of Volume I.

Definition 4.1.5. Given $C \in C(R\text{-Mod})$ and an ordinal σ, then a family $(C^\alpha | \alpha \le \sigma)$ of subcomplexes C^α of C (one for each ordinal α with $\alpha \le \sigma$) is said to be a *filtration* of C of length σ if

a) $C^\alpha \subset C^{\alpha'}$ when $\alpha \le \alpha' \le \sigma$

b) $C^\beta = \bigcup_{\alpha<\beta} C^\alpha$ when $\beta \le \sigma$ is a limit ordinal

c) $C^\sigma = C$ and $C^0 = 0$.

If $\mathcal{S}$ is a class of objects of $C(R\text{-Mod})$, then a filtration $(C^\alpha \mid \alpha \le \sigma)$ is said to be an $\mathcal{S}$*-filtration* of C if $C^{\alpha+1}/C^\alpha$ is isomorphic to an object of $\mathcal{S}$ whenever $\alpha+1 \le \sigma$. For any such class $\mathcal{S}$, by Filt($\mathcal{S}$), we mean the class of complexes $C \in C(R\text{-Mod})$ that have an $\mathcal{S}$-filtration of length σ for some ordinal σ. The class $\mathcal{S}$ is said to be closed under taking filtrations if $\text{Filt}(\mathcal{S}) = \mathcal{S}$.

With this terminology we get the basic result of Eklof.

Theorem 4.1.6. *If* $(\mathcal{A}, \mathcal{B})$ *is a cotorsion pair in* $C(R\text{-Mod})$*, then* $\text{Filt}(\mathcal{A}) = \mathcal{A}$.

Proof. This theorem was proved for modules as Corollary 7.3.5 of Volume I. □

We would like to use Theorem 4.1.6 to give necessary and sufficient conditions on a class $\mathcal{A}$ of objects of $C(R\text{-Mod})$ in order that $\mathcal{A}$ be the first component of a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $C(R\text{-Mod})$ that is cogenerated by a set. So our first necessary condition on $\mathcal{A}$ is that $\text{Filt}(\mathcal{A}) = \mathcal{A}$. We now note two other properties such an $\mathcal{A}$ must have. First if $P \in C(R\text{-Mod})$ is projective, then since $\text{Ext}^1(P, D) = 0$ for all $D \in C(R\text{-Mod})$, and so in particular $\text{Ext}^1(P, B) = 0$ for all $B \in \mathcal{B}$, we get that $P \in \mathcal{A}$. Then if $A = A_1 \oplus A_2$ for $A \in \mathcal{A}$ we have $\text{Ext}^1(A_1 \oplus A_2, D) = \text{Ext}^1(A_1, D) + \text{Ext}^1(A_2, D)$ for all $D \in C(R\text{-Mod})$. This shows $\mathcal{A}$ is closed under taking direct summands.

Now let $\mathcal{S}$ be a set of objects of $C(R\text{-Mod})$ and let $(\mathcal{A}, \mathcal{B})$ be the cotorsion pair cogenerated by $\mathcal{S}$. The proof of Theorem 7.4.1 of Volume I says that for every $C \in C(R\text{-Mod})$ there is an exact sequence $0 \to C \to B \to A \to 0$ with $B \in \mathcal{B}$ and with $A \in \text{Filt}(\mathcal{S})$. So $A \in \text{Filt}(\mathcal{S}) \subset \text{Filt}(\mathcal{A}) = \mathcal{A}$. If we then consider the proof of Proposition 7.1.7 of Volume I (with C playing the role of M), we see that for every $C \in C(R\text{-Mod})$ we have an exact sequence

$$0 \to B \to \bar{A} \to C \to 0$$

where there is an exact sequence

$$0 \to P \to \bar{A} \to A \to 0$$

with P projective and with $A \in$ Filt($\mathcal{S}$).

So if $P \in$ Filt($\mathcal{S}$), then $\bar{A} \in$ Filt($\mathcal{S}$). Note that the P of Proposition 7.1.7 of Volume I could be chosen to be a free module, and in our situation P could be chosen a free complex. But then to get $P \in$ Filt($\mathcal{S}$), we only need $S^k(\bar{R}) \in \mathcal{S}$ for each $k \in \mathbb{Z}$ (see Section 1.3).

If a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is cogenerated by a set $\mathcal{S}$, then letting $\mathcal{T}$ be $\mathcal{S}$ along with all the $S^k(\bar{R})$ for $k \in \mathbb{Z}$, then $\mathcal{T}$ is a set and cogenerates $(\mathcal{A}, \mathcal{B})$. So we could assume that for each $k \in Z$, $S^k(\bar{R}) \in \mathcal{S}$. So we get:

Theorem 4.1.7. *A cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $C(R$-Mod$)$ is cogenerated by a set if and only if there is a set $\mathcal{S}$ of objects of $C(R$-Mod$)$ such that $S^k(\bar{R}) \in \mathcal{S}$ for all $k \in \mathbb{Z}$ and such that $\mathcal{A}$ consists of all direct summands of objects of Filt($\mathcal{S}$).*

Example 4.1.8. Let $\mathcal{S}$ consist of precisely all the $S^k(\bar{R})$, $k \in \mathbb{Z}$. If we appeal to the theorem above, we see that $\mathcal{S}^\perp$ consists of all the exact complexes. We will let $\mathcal{E}$ denote the class of exact complexes. So $\mathcal{E} = \mathcal{S}^\perp$ and we get that $({}^\perp\mathcal{E}, \mathcal{E})$ is a cotorsion pair in $C(R$-Mod$)$ which is cogenerated by a set.

The complexes $P \in {}^\perp\mathcal{E}$ go by various names. We will call them *Dold projective complexes*.

The cotorsion pair $({}^\perp\mathcal{E}, \mathcal{E})$ has the interesting property that $\mathcal{E}$ is the first component of a cotorsion pair $(\mathcal{E}, \mathcal{E}^\perp)$ which is cogenerated by a set. This will follow from a later general result, but to illustrate a useful technique we will give a proof here.

We recall that for $C \in C(R$-Mod$)$, we define the *cardinality* of C (denoted $|C|$) to be $\sum_{n\in\mathbb{Z}} |C_n|$.

Lemma 4.1.9. *Let R be a ring and κ be an infinite cardinal such that $|R| \le \kappa$. Let $E \in \mathcal{E}$ and let $x \in E$ (so $x \in E_n$ for a unique $n \in \mathbb{Z}$). Then there exists an exact subcomplex $E' \subset E$ with $x \in E'$ and such that $|E'| \le \kappa$.*

Proof. We will use what is called a *zig-zag technique* to construct E'. We first note that since $|R| \le \kappa$ and since K is infinite, if $X \subset M$ for $M \in R$-Mod where X is a subset with $|X| \le \kappa$, then if S is the submodule of M generated by X we have $|S| \le \kappa$.

We will construct E' as the union of an increasing sequence of subcomplexes.

$$C^0 \subset C^1 \subset C^2 \subset \cdots$$

of E where $x \in C^0$, $|C^n| \le K$ and $Z(C^n) \subset B(C^{n+1})$ for each $n \in \mathbb{Z}$. Then if $E' = \bigcup_{n\in\mathbb{Z}} C^n$, we have $Z(E') = \bigcup_{n\in\mathbb{Z}} Z(C^n) \subset \bigcup_{n\in\mathbb{Z}} B(C^{n+1}) \subset B(E')$. So

$Z(E') = B(E')$ and E' will be exact. So we have $x \in C^0 \subset E'$ and $|E'| \leq \kappa$. Letting $x \in E_k$ and C^0 be such that $C^0_k = R$ and $C^0_{k-1} = d_k(Rx)$. Then set $C^0_n = 0$ if $n \neq k, k-1$. So $C^0 \subset E$ is a subcomplex, $x \in C^0$ and $|C^0| \leq \kappa$. Having constructed C^n with $|C^n| \leq \kappa$ we want to construct C^{n+1} with $C^n \subset C^{n+1}$ but such that $Z(C^n) \subset B(C^{n+1})$ and with $|C^{n+1}| \leq \kappa$. For each $\ell \in \mathbb{Z}$ we have

$$Z_\ell(C^n) \subset Z_\ell(E) = B_\ell(E).$$

So since we assumed $|R| \leq \kappa$, we can find a submodule $S_{\ell+1} \subset E_{\ell+1}$ such that $Z_\ell(C^n) \subset d_{\ell+1}(S_{\ell+1})$.

We now define C^{n+1}_ℓ so that for $\ell \in \mathbb{Z}$, $C^{n+1}_\ell = C^n_\ell + S_\ell + d_{\ell+1}(S_{\ell+1})$. Then C^{n+1} is a subcomplex of E, $C^n \subset C^{n+1}$ and by construction $Z(C^n) \subset B(C^{n+1})$. So finally we have the desired subcomplex $E' \subset E$. □

Remark 4.1.10. We will have other occasions when we need to use the technique we used in this proof. When we do so we will call it the *zig-zag technique*. So we will allow ourselves to appeal to the technique without giving details.

With the notation of the lemma we have $E' \subset E$ with both E' and exact. Hence E/E' is exact.

Definition 4.1.11. Let $\mathcal{A}$ be a class of objects of $C(R\text{-Mod})$. Then $\mathcal{A}$ is said to be a *Kaplansky class* if there is a cardinal number κ such that when we have $x \in \mathcal{A}$ for $\mathcal{A} \in \mathcal{A}$, there is a subcomplex $\mathcal{A}' \subset \mathcal{A}$ with $x \in A'$ and with A' and $A/A' \in \mathcal{A}$ where $|A'| \leq \kappa$. We also say $\mathcal{A}$ is *Kaplansky* relative to κ.

So Lemma 4.1.9 could be rephrased. It says that the class $\mathcal{E}$ of exact complexes in $C(R\text{-Mod})$ is Kaplansky relative to any infinite cardinal κ such that $|R| \leq \kappa$. We can use the class $\mathcal{E}$ to illustrate another important notion. Again with $|R| \leq K$, K an infinite cardinal, let $E \in \mathcal{E}$. Let $E^0 = 0 \subset E$. For any $x \in E$, let $E^1 \subset E$ be exact with $x \in E^1$ and $|E^1| \leq \kappa$ (by Lemma 4.1.9). Then considering $E/E^1 \in \mathcal{E}$ and some $y \in E/E^1$, we can find $E^2/E^1 \subset E/E^1$ with $y \in E^2/E^1$, $E^2/E^1 \in \mathcal{E}$ and $|E^2/E^1| \leq \kappa$. But then is clear E^2 is exact and we can repeat the procedure with E/E^2. Then noting that the union of any chain of exact subcomplexes of E is exact, we see that for some ordinal number σ we can construct a filtration $(E^\alpha \mid \alpha \leq \sigma)$ of E such that when $\alpha + 1 \leq \sigma$ we have $E^{\alpha+1}/E^\alpha$ exact (i.e. $E^{\alpha+1}/E^\alpha \in \mathcal{E}$ and $|E^{\alpha+1}/E^\alpha| \leq \kappa$.

Definition 4.1.12. If $\mathcal{A}$ is a class of objects of $C(R\text{-Mod})$, we say that $\mathcal{A}$ is *deconstructible* if there is a cardinal number κ such that every $A \in \mathcal{A}$ has a filtration $(A^\alpha \mid \alpha \leq \sigma)$ for some ordinal σ such that when $\alpha + 1 \leq \sigma$ we have $A^{\alpha+1}/A^\alpha \in \mathcal{A}$ and $|A^{\alpha+1}/A^\alpha| \leq \kappa$. In this case we also say that $\mathcal{A}$ is deconstructible relative to κ.

So with the same $\mathcal{E}$ we have that $\mathcal{E}$ is deconstructible relative to any infinite κ with $\kappa \geq |R|$.

Let $\mathcal{A}$ be a class which is deconstructible relative to the cardinal number κ. Then there is a subset $\mathcal{S} \subset \mathcal{A}$ of representatives of $A \in \mathcal{A}$ with $|A| \leq \kappa$. Then by the definition of deconstructibility, we have $\mathcal{A} \subset \text{Filt}(\mathcal{S})$.

Theorem 4.1.13. *Suppose $\mathcal{A}$ is a class of objects of $C(R\text{-Mod})$ such that*

a) $\mathcal{A} = \text{Filt}(\mathcal{A})$

b) $\mathcal{A}$ *is closed under taking direct summands*

c) $\mathcal{A}$ *is deconstructible*

d) $S^k(\bar{R}) \in \mathcal{A}$ *for every* $k \in \mathbb{Z}$.

Then with $\mathcal{B} = \mathcal{A}^\perp$, $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair which is cogenerated by a set.

Proof. This follows from Theorem 4.1.7. We choose $\mathcal{S} \subset \mathcal{A}$ as a set including representatives of $A \in \mathcal{A}$ with $|A| \leq \kappa$ where $\mathcal{A}$ is deconstructible relative to K, but also requiring that $S^k(\bar{R}) \in \mathcal{A}$ for each $k \in \mathbb{Z}$. Then $\mathcal{A} \subset \text{Filt}(\mathcal{S})$. But since $\mathcal{S} \subset \mathcal{A}$, $\text{Filt}(\mathcal{S}) \subset \text{Filt}(\mathcal{A}) = \mathcal{A}$. So $\mathcal{A} = \text{Filt}(\mathcal{S})$. Since $\mathcal{A}$ is closed under direct summands we get that $\mathcal{A}$ consists of all direct summands of objects of $\text{Filt}(\mathcal{S})$. So now Theorem 4.1.7 gives us the result. □

There is a module theoretic version Theorem 4.1.13. We will state it here.

Theorem 4.1.14. *Suppose $\mathcal{A}$ is a class of objects of R-Mod such that*

a) $\mathcal{A} = \text{Filt}(\mathcal{A})$

b) $\mathcal{A}$ *is closed under direct summand*

c) $\mathcal{A}$ *is deconstructible*

d) $R \in \mathcal{A}$.

Then with $\mathcal{B} = \mathcal{A}^\perp$, the pair $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair which is cogenerated by a set.

Corollary 4.1.15. *With $\mathcal{E}$ the class of exact complexes in $C(R\text{-Mod})$, $(\mathcal{E}, \mathcal{E}^\perp)$ is a cotorsion pair in $C(R\text{-Mod})$ which is cogenerated by a set.*

Proof. We have already noted that $\mathcal{E}$ satisfies a), b), and c) of the theorem above. We get $S^k(\bar{R}) \in \mathcal{E}$ since $S^k(\bar{R})$ is exact. So $\mathcal{E}$ satisfies a)–d) and we have established the claim. □

4.2 Cotorsion Triplets

In the last section we argued that with $\mathcal{E}$ the class of exact complexes in $C(R\text{-Mod})$, we have that $({}^{\perp}\mathcal{E}, \mathcal{E})$ and $(\mathcal{E}, \mathcal{E}^{\perp})$ are both cotorsion pairs which are cogenerated by sets. So this motivates the next definition.

Definition 4.2.1. If $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are each classes of objects of $C(R\text{-Mod})$, we say $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a *cotorsion triplet* in $C(R\text{-Mod})$ if each of $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{B}, \mathcal{C})$ are cotorsion pairs in $C(R\text{-Mod})$. We say the triple is *complete* if each of the pairs $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{B}, \mathcal{C})$ is complete.

In Section 4.1 we showed that $({}^{\perp}\mathcal{E}, \mathcal{E}, \mathcal{E}^{\perp})$ is a complete cotorsion triplet in $C(R\text{-Mod})$. Another complete triplet is $({}^{\perp}C(R\text{-Mod}), C(R\text{-Mod}), C(R\text{-Mod})^{\perp})$ where ${}^{\perp}C(R\text{-Mod})$ consists of the projective complexes and $C(R\text{-Mod})^{\perp}$ consists of the injective complexes.

We now give a method for constructing a family of complete triplets.

Definition 4.2.2. A complex $P \in C(R\text{-Mod})$ is said to be a *perfect complex* if $P_n = 0$ except for a finite number of $n \in Z$ and if each P_n is a finitely generated projective module.

If a complex P is perfect and exact then it is easy to see that P is projective. If $f : P \to Q$ is a morphism of perfect complexes. Then $C(f)$ is perfect. If moreover $f : P \to Q$ is a homology isomorphism then $C(f)$ is exact (Proposition 2.4.2). So in this case $C(f)$ is a projective complex. In particular $C(\mathrm{id}_p)$ is projective for any perfect complex P. We have the exact sequence

$$0 \to P \to C(f) \to S(P) \to 0$$

and the sequence

$$0 \to S^{-1}P \to S^{\perp}C(f) \to P \to 0.$$

is exact. This is a partial project resolution of P. So applying S^{-1} to this sequence we get the exact sequence

$$0 \to S^{-2}P \to S^{-2}C(f) \to S \to P \to 0.$$

So we see that for any $n \geq 1$ we get a partial projective resolution

$$0 \to S^{-n}P \to S^{-n}C(f) \to S^{-n+1}C(f) \to \cdots \to S^{-1}C(f) \to P \to 0$$

of P. Hence we get that for $k \geq 1$ and $n \geq 1$,

$$\mathrm{Ext}^k(S^{-n}P, D) = \mathrm{Ext}^{k+n}(P, D)$$

for any complex D. So we get the next result.

Lemma 4.2.3. *If $P \in C(R\text{-Mod})$ is a perfect complex, then for any complex D and $k \geq 1$ and $n \geq 1$ we get that*

$$\mathrm{Ext}^k(S^{-n}P, D) = \mathrm{Ext}^{k+n}(P, D).$$

Lemma 4.2.4. *If $P \in C(R\text{-Mod})$ is a perfect complex, then $\mathrm{Ext}^1(P, Q) = 0$ for any projective complex Q.*

Proof. By Proposition 2.1.4, we have $\mathrm{Ext}^1(P, S^k(\bar{R})) \cong \mathrm{Ext}^1(P_{k-1}, R)$ for any $k \in \mathbb{Z}$. Since P_{k-1} is projective, we get $\mathrm{Ext}^1(P_{k-1}, R) = 0$ and so $\mathrm{Ext}^1(P, S^k(\bar{R})) = 0$. So now using the fact that P is a finitely generated complex, we get that $\mathrm{Ext}^1(P, -)$ commutes with direct sums. Hence $\mathrm{Ext}^1(P, F) = 0$ for any free complex F (see Section 1.3) and so since any projective complex Q is a direct summand of a free complex F we also get $\mathrm{Ext}^1(P, Q) = 0$. $\square$

Proposition 4.2.5. *Let $\mathcal{S}$ be a set of perfect complexes in $C(R\text{-Mod})$. Let $\mathcal{B}$ consist of all complexes B such that $\mathrm{Ext}^n(P, B) = 0$ for all $P \in \mathcal{G}$ for all $n \geq 1$. Then there is a complete cotorsion triplet $(\mathcal{A}, \mathcal{B}, \mathcal{C})$. In fact $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{B}, \mathcal{C})$ are cogenerated by sets and $(\mathcal{P}, \mathcal{C})$ is perfect.*

Proof. If we let $\mathcal{T}$ consist of all $S^{-n}P$ for $P \in \mathcal{S}$ and $n \geq 0$, then $\mathcal{T}$ is a set and by the observation above we have $\mathcal{T}^{\perp} = \mathcal{B}$.

Note that by the definition of $\mathcal{T}$, we have that for $Q \in \mathcal{T}$, $B \in \mathcal{B}$, $\mathrm{Ext}^n(Q, B) = 0$ for all $n \geq 1$.

Now if we let $\mathcal{A} = {}^{\perp}\mathcal{B}$, then $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair which is cogenerated by a set. We now prove that $\mathcal{B}$ satisfies a), b) c) and d) of Theorem 4.1.13. Since $\mathcal{B} = \mathcal{T}^{\perp}$ we get b). To get d), we will appeal to Proposition 2.1.4. By that result we get $\mathrm{Ext}^1(Q, S^k(\bar{R})) \cong \mathrm{Ext}^1(Q_{k-1}, R)$, for any $Q \in \mathcal{T}$. By the definition of $\mathcal{T}$, Q_{k-1} is projective. So $\mathrm{Ext}^1(Q_{k-1}, R) = 0$ and so $\mathrm{Ext}^1(Q, S^k(\bar{R})) = 0$. So for any $k \in \mathbb{Z}$, $S^k(\bar{R}) \in Q^1 = \mathcal{B}$. To get a), i.e. $\mathcal{B} = \mathrm{Filt}(\mathcal{B})$, let $(B^\alpha \mid \alpha \leq \sigma)$ be a filtration of B where $B^{\alpha+1}/B^\alpha \in \mathcal{B}$ whenever $\alpha + 1 \leq \sigma$. We want to argue that $B \in \mathcal{B}$.

We argue by transfinite induction that $B^\alpha \in \mathcal{B}$. If $\alpha = 0$, $B^0 = 0$ and so trivially $B^0 \in \mathcal{B}$. If $B^\alpha \in \mathcal{B}$ with $\alpha < \sigma$ then $B^{\alpha+1}/B^\alpha \in \mathcal{B}$. But we have the exact $0 \to B^\alpha \to B^{\alpha+1} \to B^{\alpha+1}/B^\alpha \to 0$. So if $P \in \mathcal{S}$ we have the exact $0 = \mathrm{Ext}^1(P, B^\alpha) \to \mathrm{Ext}^1(P, B^{\alpha+1}) \to \mathrm{Ext}^1(P, B^{\alpha+1}/B^\alpha) = 0$ and so $\mathrm{Ext}^1(P, B^{\alpha+1}) = 0$. Hence $B^{\alpha+1} \in \mathcal{B}$.

Now if $\beta \leq \sigma$ is a limit ordinal and if $B^\alpha \in \mathcal{B}$ for all $\alpha < \beta$ we want to argue that $B^\beta \in \mathcal{B}$, that is $\mathrm{Ext}^1(P, B^\beta) = 0$ for all $P \in \mathcal{S}$.

Since we have the partial projective resolution

$$0 \to S^{-1}P \to S^{-1}C(f) \to P \to 0,$$

we can complete $\mathrm{Ext}^1(P, B^\beta)$ using this resolution.

The complex $S^{-1}P$ is finitely generated and so any morphism $S^{-1}P \to B^{\beta} = \bigcup_{\alpha<\beta} \beta^{\alpha}$ will have its image in B^{α} for some $\alpha < \beta$. Since we assumed $B^{\alpha} \in \mathcal{B}$, that is $\mathrm{Ext}^1(P, \beta^{\alpha}) = 0$ when $P \in \mathcal{S}$, we get that $S^{-1}P \to B^{\alpha}$ has an extension $S^{-1}C(f) \to B^{\alpha}$. So the original $S^{-1}P \to B^{\beta}$ has an extension $S^{-1}C(f) \to B^{\beta}$. So we get $\mathrm{Ext}^1(P, B^{\beta}) = 0$. This completes the proof. □

4.3 The Dold Triplet

In Example 4.1.8, we let $\mathcal{S}$ consist of all the complexes $S^k(\bar{R})$ for all $k \in \mathbb{Z}$. These complexes are all perfect. So we can apply Proposition 4.2.5 and get a cotorsion triplet. By Corollary 2.1.7 we have that $\mathcal{S}^{\perp} = \mathcal{E}$ where $\mathcal{E}$ is the class of exact complexes. So our triplet in this case is $({}^{\perp}\mathcal{E}, \mathcal{E}, \mathcal{E}^{\perp})$. We will call the $P \in {}^{\perp}\mathcal{E}$ the *Dold projective complexes* and the $I \in \mathcal{E}^{\perp}$ the *Dold injective complex*.

Proposition 4.3.1. *The complex $P \in C(R\text{-Mod})$ is a Dold projective complex if and only if P_n is a projective module for every $n \in \mathbb{Z}$ and if every morphism $f : P \to E$ where E is an exact complex is homotopic to* 0.

Proof. For every $N \in R$-Mod, $S^k(\bar{N})$ is an exact complex. By Proposition 2.1.4, $\mathrm{Ext}^1(P, S^k(\bar{N})) = \mathrm{Ext}^1(P_{k-1}, N)$. So if $P \in {}^{\perp}\mathcal{E}$, we get that $\mathrm{Ext}^1(P_{k-1}, N) = 0$. So for every $k \in \mathbb{Z}$, P_{k-1} is projective. Now given $f : P \to E$ we have the exact sequence $0 \to E \to C(f) \to S(P) \to 0$ and so the exact $0 \to S^{-1}E \to S^{-1}C(f) \to P \to 0$. But $S^{-1}E$ is also exact. Since we assume $P \in {}^{\perp}\mathcal{E}$, this sequence is split exact. So $0 \to E \to C(f) \to S(P) \to 0$ is split exact. By Proposition 3.3.2 this means $f : P \to E$ is homotopic to 0.

Conversely assume P is such that each P_n is projective and that each $f : P \to E$ with E exact is homotopic to 0. We want to argue that for all exact E, $\mathrm{Ext}^1(P, E) = 0$. So this means that every short exact sequence $0 \to E \to U \to P \to 0$ splits. Since each P_n is projective we get that $0 \to E \to U \to P \to 0$ splits at the module level. That is, for each $n \in \mathbb{Z}$, $0 \to E_n \to U_n \to P_n \to 0$ is split exact. But as observed in Section 3.3, this means that the sequence is isomorphic to a sequence of the form

$$0 \to E \to C(g) \to P \to 0$$

where $g : S^{-1}P \to E$ is a morphism of complexes. By hypothesis $S(g) : P \to S(E)$ is homotopic to 0. Hence $g : S^{-1}P \to E$ is homotopic to 0 and $0 \to E \to C(g) \to P \to 0$ is split exact by Proposition 3.3.2. So $0 \to E \to U \to P \to 0$ is split exact. Since we now have $\mathrm{Ext}^1(P, E) = 0$ for all exact E, we have $P \in {}^{\perp}\mathcal{E}$. That is, P is Dold projective. □

Dual arguments give us the next result.

Proposition 4.3.2. *The complex* $I \in C(R\text{-Mod})$ *is Dold injective if and only if* I_n *is a injective module for every* $n \in \mathbb{Z}$ *and if every morphism* $g : E \to I$ *where* E *is an exact complex is homotopic to* 0.

4.4 More on Cotorsion Pairs and Triplets

Definition 4.4.1. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $C(R\text{-Mod})$ is said to be *hereditary* if $\text{Ext}^n(A, B) = 0$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $n \geq 1$.

Proposition 4.4.2. *If* $(\mathcal{A}, \mathcal{B})$ *is a cotorsion pair in* $C(R\text{-Mod})$*, then the following are equivalent:*

a) $(\mathcal{A}, \mathcal{B})$ *is hereditary*

b) *If* $A \in \mathcal{A}$ *and if* $0 \to A' \to P \to A \to 0$ *is exact where* P *is a projective complex, then* $A' \in \mathcal{A}$

c) *If* $B \in \mathcal{B}$ *and* $0 \to B \to I \to B' \to C$ *is exact where* I *is an injective complex, then* $B' \in \mathcal{B}$.

Proof. Establishing the equivalence of a), b) and c) is just a matter of dimension shifting. To indicate how the argument goes we prove a) implies b). So with $A \in \mathcal{A}$ and $0 \to A' \to P \to A \to 0$ exact where P is projective, let $B \in \mathcal{B}$. Then we get an exact sequence $\text{Ext}^1(P, B) \to \text{Ext}^1(A', B) \to \text{Ext}^2(A, B)$. But $\text{Ext}^1(P, B) = 0$ since P is projective and $\text{Ext}^2(A, B) = 0$ by hypothesis. Hence $\text{Ext}^1(A', B) = 0$. The reader can now argue that the other implications hold. □

Definition 4.4.3. A cotorsion triplet $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ in $C(R\text{-Mod})$ is said to be *hereditary* if each of the cotorsion pairs $(\mathcal{A}, B)$ and $(\mathcal{B}, \mathcal{C})$ are hereditary.

Our next result will show how important this concept can be. First we recall some results and terminology. Suppose $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion pair in $C(R\text{-Mod})$, then for $C \in C(R\text{-Mod})$, we have an exact sequence $0 \to B \to A \to C \to 0$. Then if $A' \in \mathcal{A}$ we have the exact

$$\text{Hom}(A', A) \to \text{Hom}(A, C) \to \text{Ext}^1(A', B) = 0,$$

i.e. $\text{Hom}(A', A) \to \text{Hom}(A, C) \to 0$ is exact. So using the terminology of Chapter 5 of Volume I (but applied to complexes), we have the $A \to C$ is an $\mathcal{A}$-precover of C. Using this procedure we can construct a complex (of complexes)

$$\cdots \to A^{-2} \to A^{-1} \to A^0 \to C \to 0$$

such that if $A \in \mathcal{A}$ then $\text{Hom}(A, -)$ applied to this complex gives an exact sequence. Using the terminology of Chapter 8 of Volume I, this is a left $\mathcal{A}$-resolution of C. As noted in Section 8.1 of Volume I, the associated complex

$$\cdots \to A^{-2} \to A^{-1} \to A^0 \to 0$$

is unique up to a homotopy isomorphism. In fact the complex

$$\cdots \to A^{-2} \to A^{-1} \to A^0 \to C \to 0$$

is also unique up to a homotopy isomorphism.

We will freely use other terminology of Volume I but adapted to the categories $C(R\text{-Mod})$.

Theorem 4.4.4. *Let* $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ *be a cotorsion triplet in* $C(R\text{-Mod})$*. If* $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ *is hereditary, then* $\mathcal{A} \cap \mathcal{B}$ *is the class of projective complexes and* $\mathcal{B} \cap \mathcal{C}$ *is the class of injective complexes. If* $(\mathcal{A}, B, \mathcal{C})$ *is hereditary and complete, then* $\mathrm{Hom}(-,-)$ *is right balanced by* $\mathcal{A} \times \mathcal{C}$ *(see Definition* 8.2.13 *of Volume* I*).*

Proof. We assume $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is hereditary and let $D \in \mathcal{A} \cap \mathcal{B}$. We want to prove that D is projective. Let $0 \to K \to P \to D \to 0$ be exact with P projective. Since $D \in \mathcal{B}$ and since $(\mathcal{B}, \mathcal{C})$ is hereditary, $K \in \mathcal{B}$. But then since $D \in \mathcal{A}$ and since $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair, $\mathrm{Ext}^1(D, K) = O$. So the sequence $0 \to K \to P \to D \to 0$ is split exact. Hence D is projective. Conversely, if $P \in C(R\text{-Mod})$ is projective then since $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair then $A \in \mathcal{A}$. But $(\mathcal{B}, \mathcal{C})$ is also a cotorsion pair and so $P \in \mathcal{B}$. Hence $P \in \mathcal{A} \cap \mathcal{B}$. Thus $\mathcal{A} \cap \mathcal{B}$ is the class of projective complexes.

A dual argument gives that $\mathcal{B} \cap \mathcal{C}$ is the class of injective complexes when $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a hereditary triplet. Now we assume that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is complete and hereditary.

Given $D \in C(R\text{-Mod})$, we use the fact that $(\mathcal{A}, B)$ is complete to construct a complex

$$\cdots \to A^{-1} \to A^0 \to D \to 0$$

so that if $B^{-1} = \mathrm{Ker}(A^{-n+1} \to A^{-n+2})$ where $n \geq 2$, then $B^{-n} \in \mathcal{B}$ for $n \geq 1$ and $0 \to B^{-1} \to A^0 \to D \to 0$ and $0 \to B^{-n} \to A^{-n+1} \to B^{-n+1} \to 0$ for $n \geq 2$ are exact. Let $0 \to B^{-1} \to E \to B' \to 0$ be exact with E injective. Since $(\mathcal{A}, \mathcal{B})$ is hereditary, $B' \in \mathcal{B}$ and so $\mathrm{Ext}^1(B', C) = 0$ since $(\mathcal{B}, \mathcal{C})$ is a cotorsion pair. This gives that $\mathrm{Hom}(E, C) \to \mathrm{Hom}(B^{-1}, C) \to 0$ exact, and so that the map $B^{-1} \to C$ has an extension $E \to C$.

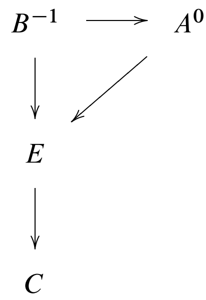
Now since E is injective and $0 \to B^{-1} \to A^0$ is exact we get that the $B^{-1} \to E$ can be extended to $A^0 \to E$. We want to prove that if $B \in \mathcal{B}$, then $\mathrm{Hom}(-, B)$ leaves the sequence exact. We argue that it leaves the sequence $0 \to B^{-1} \to A^0 \to D \to 0$ exact. Then same argument will give that it leaves each $0 \to B^{-n} \to B^{-n+1} \to B^{-n+2} \to 0$ exact and so that it leaves

$$\cdots \to A^{-2} \to A^{-1} \to A^0 \to D \to 0$$

exact.

Let $B^{-1} \to C$ be given. We want to argue it has an extension $A^0 \to C$.

But we have a commutative diagram



So we have the desired extension $A^0 \to C$.

A dual argument gives the rest of the proof that $\mathrm{Hom}(-,-)$ is right balanced by $\mathcal{A} \times \mathcal{C}$. □

Proposition 4.3.2 guarantees that for each set $\mathcal{S}$ of perfect complexes in $C(R\text{-Mod})$, we get a complete cotorsion triplet $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ with $\mathcal{B}$ consisting of all B such that $\mathrm{Ext}^n(P, B) = 0$ for all $n \geq 1$ and all $P \in \mathcal{S}$. In fact $\mathcal{B} = \mathcal{T}^{\perp}$ where $\mathcal{T}$ consists of all $S^{-n}P$ for $P \in \mathcal{S}$ and $n \geq 0$. We would like to also generate triplets $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ that are hereditary. The next result allows us to do this.

Proposition 4.4.5. *If $\mathcal{S}$ is a set of perfect complexes and if $\mathcal{U}$ is the set of all the complexes $S^k(P)$ where $P \in \mathcal{S}$, $k \in \mathbb{Z}$, then there is a complete and hereditary cotorsion triplet $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ with $\mathcal{B} = \mathcal{U}^{\perp}$.*

Proof. We apply Proposition 4.3.2 but using $\mathcal{U}$ instead of $\mathcal{S}$. Then by the proof of that Proposition, we get a complete triplet $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ with $\mathcal{B} = \mathcal{U}^{\perp}$. We also get that $(\mathcal{A}, \mathcal{B})$ is hereditary. So we only need argue that $(\mathcal{B}, \mathcal{C})$ is hereditary. By Proposition 4.4.2, it suffices to show that if $0 \to B' \to Q \to B \to 0$ is exact with $B \in \mathcal{B}$ and Q projective, then $B' \in \mathcal{B}$. So we need to show that $\mathrm{Ext}^1(P, B') = 0$ for all $P \in \mathcal{U}$. But we have the exact sequence

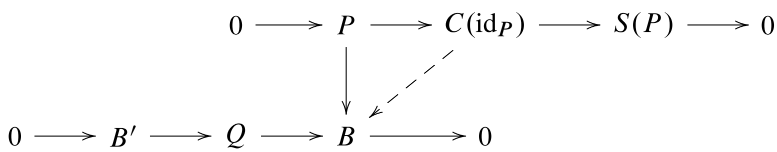
$$\mathrm{Hom}(P, Q) \to \mathrm{Hom}(P, B) \to \mathrm{Ext}^1(P, B') \to \mathrm{Ext}^1(P, Q)$$

So to get $\mathrm{Ext}^1(P, B') = 0$ it suffices to show that $\mathrm{Ext}^1(P, Q) = 0$ and that

$$\mathrm{Hom}(P, Q) \to \mathrm{Hom}(P, B) \to 0$$

is exact. By we have $\mathrm{Ext}^1(P, Q) = 0$.

To get that $\mathrm{Hom}(P, Q) \to \mathrm{Hom}(P, B) \to 0$ is exact, we consider the diagram



with exact rows. Since $P \in \mathcal{U}$, we have $S(P) \in \mathcal{U}$ and so $\mathrm{Ext}^1(S(P), B) = 0$. So this gives an extension $C(\mathrm{id}_P) \to B$ of $P \to B$. But $C(\mathrm{id}_P)$ is a projective complex and so $C(\mathrm{id}_P) \to B$ has a lifting $C(\mathrm{id}_P) \to Q$. Restricting to P, we get a lifting $P \to Q$ of the original $P \to B$. □

Remark 4.4.6. We have a plentiful supply of hereditary triplets in $C(R\text{-Mod})$. So it is natural to ask if we have examples of hereditary quadruplets $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ in $C(R\text{-Mod})$. But then since $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and $(B, \mathcal{C}, \mathcal{D})$ are hereditary triplets, we get that $\mathcal{B} \cap \mathcal{C}$ is both the class of projective complexes and the class of injective complexes. This means that a complex is projective if and only if it is an injective complex. In this case a module is injective if and only if it projective. Recall that in R-Mod we have the cotorsion pairs (Proj, R-Mod) (R-Mod, Inj) where Proj and Inj are the classes of projective and injective classes. So if Proj $=$ Inj we have the quadruplet (Proj, R-Mod, Proj, R-Mod) and we likewise get a quintuplet, sextuplet etc.

4.5 Exercises

1. Consider the perfect complex $K(r) = K$ of Section 3.5 (so $r \in R$ and R is commutative.) Let $\mathcal{S} = \{K\}$ in Proposition 4.4.5. Then identify the triplet $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ that we get both when $r = 0$ and when $r = 1$.
2. Let $P \in C(R\text{-Mod})$ be such that P_n is a projective module for every n and such that for some n_0, $P_n = 0$ for all $n > n_0$. Prove that P is Dold projective.
   Hint: Use Proposition 4.3.1.
3. State and prove a dual result for $I \in C(R\text{-Mod})$.
4. Argue that with $R = \mathbb{Z}$ there are an infinite number of distinct complete and hereditary cotorsion triplets in $C(\mathbb{Z}\text{-Mod})$.

Chapter 5
Adjoint Functors

In this chapter we will give a brief introduction to adjoint functors. We will give the version that suits the applications we have in mind.

5.1 Adjoint Functors

Definition 5.1.1. If $\mathcal{C}$ and $\mathcal{D}$ are categories and $S : \mathcal{C} \to \mathcal{D}$ and $T : \mathcal{D} \to \mathcal{C}$ are functors, we say that S is a *left adjoint* of T or that T is a *right adjoint* of S if the two functors $(S, Y) \mapsto \mathrm{Hom}(S(X), Y)$ and $(X, Y) \mapsto \mathrm{Hom}(X, T(Y))$ from $\mathcal{C}^0 \times \mathcal{D}$ into Sets (where Sets is the category of sets) are naturally isomorphic (see Section 1.3 of Volume I).

We note that we are using the common convention where a functor is described by what it does to objects with the assumption that it is clear what it does to morphisms.

Given a functor $S : \mathcal{C} \to \mathcal{D}$, we can ask whether S admits a right adjoint $T : D \to \mathcal{D}$.

Proposition 5.1.2. *Let $S : \mathcal{C} \to D$ be a functor. Suppose that for every object Y of $\mathcal{D}$ there exists an object X of $\mathcal{D}$ and a morphism $\sigma : S(X) \to Y$ in $\mathcal{D}$ with the following universal property: if X' is any object of $\mathcal{D}$ and if $\sigma' : S(X') \to Y$ is another morphism in $\mathcal{D}$, then there is a unique morphism $f : X' \to X$ in $\mathcal{C}$ such that $\sigma \circ S(f) = \sigma'$. Then S has a right adjoint $T : \mathcal{D} \to \mathcal{C}$.*

Proof. We give a sketch of a proof. We first note that if $\bar{\sigma} : S(X) \to Y$ is another morphism in $\mathcal{D}$ (where X is an object of $\mathcal{C}$) having the same universal property as $\sigma : S(X) \to Y$, then there exist unique morphisms $f : X \to \bar{X}$ and $g : \bar{X} \to X$ such that $\bar{\sigma} \circ S(f \circ g) = \bar{\sigma} \circ S(f) \circ S(g) = \bar{\sigma}$. But $\bar{\sigma} \circ S(\mathrm{id}_X) = \bar{\sigma}$. So we have $f \circ g = \mathrm{id}_{\bar{X}}$. Similarly we get $g \circ f = \mathrm{id}_X$. Hence f and g are isomorphisms and $f^{-1} = g$.

This shows that the object X such that $\sigma : S(X) \to Y$ has our universal property is unique up to isomorphism.

Suppose that for each object Y of $\mathcal{D}$ we choose one such X and that we let $T(Y)$ denote the X that we choose. To get a corresponding functor $T : \mathcal{D} \to \mathcal{C}$, we need to give $T(h)$ for a morphism $h : Y_1 \to Y_2$. Let $\sigma_1 : S(T(Y_1)) \to Y_1$, and

$\sigma_2 : S(T(Y_2)) \to Y_2$ be the given universal morphisms. Then considering the diagram

$$\begin{array}{ccc} S(T(Y_1)) & \xrightarrow{\sigma_1} & Y_1 \\ \downarrow & & \downarrow h \\ S(T(Y_2)) & \xrightarrow{\sigma_2} & Y_2 \end{array}$$

and using the universal property of σ_2, we get that there is a unique morphism $f : T(Y_1) \to T(Y_2)$ such that $S(f) : S(T(Y_1)) \to S(T(Y_2))$ make the diagram above commutative. We then let $T(h) = f$ for this unique f.

Then it can be quickly checked that $T : \mathcal{D} \to \mathcal{C}$ is a functor and that T is a right adjoint of S. □

Remark 5.1.3. Proposition 5.1.2 gives a condition on $S : \mathcal{C} \to \mathcal{D}$ that is sufficient to guarantee that S has a right adjoint $T : \mathcal{D} \to \mathcal{C}$. In fact the condition is necessary and every right adjoint T of S can be constructed as in the proof of the Proposition above with this observation it can then be seen that T is unique up to isomorphism.

If we consider the special case where $\mathcal{C}$ is some full subcategory of $\mathcal{D}$ and where $S : \mathcal{C} \to \mathcal{D}$ is just the embedding functor, then a $\sigma : S(X) = X \to Y$ with the universal property is a $\mathcal{C}$-precover of Y (see Section 5.1 of Volume I). In fact it is such that if $\sigma' : X' \to Y$ is another morphism of $\mathcal{D}$ with X' in $\mathcal{C}$, then there is a unique morphism $f : X' \to X$ so that $\sigma \circ f = \sigma'$. So this $\sigma : X \to Y$ will be a $\mathcal{C}$-cover with this unique factorization property. This observation and the duality between covers and envelopes suggests a result dual to Proposition 5.1.2.

Proposition 5.1.4. *Let $T : \mathcal{D} \to \mathcal{C}$ be a functor. Suppose that for every object X of $\mathcal{C}$ there exists an object Y of $\mathcal{D}$ and a morphism $\tau : X \to T(Y)$ in $\mathcal{C}$ with the following universal property: if Y' is any object of $\mathcal{D}$ and if $\tau' : X \to T(Y')$ is another morphism in $\mathcal{C}$, then there is a unique morphism $g : Y \to Y'$ in $\mathcal{D}$ such that $T(g) \circ \tau = \tau'$. Then T has a left adjoint $S : \mathcal{C} \to \mathcal{D}$.*

Proof. The proof is dual to the proof of Proposition 5.1.2 above. □

Now let $(\mathcal{A}, \mathcal{B})$ be a complete cotorsion pair in $C(R\text{-Mod})$. Then we can think of $\mathcal{A}$ as a full subcategory of $C(R\text{-Mod})$. So we have the embedding functor $S : \mathcal{A} \to C(R\text{-Mod})$. So we can ask whether S has a right adjoint. In general this will not be the case. However, since $(\mathcal{A}, \mathcal{B})$ is complete, for every $C \in C(R\text{-Mod})$ we will have an exact sequence

$$0 \to B \to A \xrightarrow{\sigma} C \to 0$$

with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. But this means that $\sigma : A \to C$ will be an $\mathcal{A}$-precover. But in general we will not be able to find such an $A \to C$ that has the desired universal property, i.e. so that if $\sigma' : A' \to C$ is a morphism in $C(R\text{-Mod})$ with $A' \in \mathcal{A}$ then there is a unique $f : A' \to A$ so that $\sigma \circ f = \sigma'$. To remedy this situation we go to the homotopy category $K(R\text{-Mod})$ of Chapter 3.

Definition 5.1.5. If $\mathcal{A}$ is a full subcategory of $C(R\text{-Mod})$, we let $K(\mathcal{A})$ be the full subcategory of $K(R\text{-Mod})$ having the same objects as $\mathcal{A}$.

Definition 5.1.6. If $\mathcal{A}$ is a full subcategory of $C(R\text{-Mod})$, we let $S(\mathcal{A})$ be the full subcategory of $C(R\text{-Mod})$ whose objects are the $S(A)$ where $A \in \mathcal{A}$. We will say $\mathcal{A}$ is closed under suspensions if $S(\mathcal{A}) \subset \mathcal{A}$, i.e. is $S(A) \in \mathcal{A}$ for all $A \in \mathcal{A}$. We now give the main result of this chapter.

Theorem 5.1.7. *Let* $(\mathcal{A}, \mathcal{B})$ *be a complete cotorsion pair in* $C(R\text{-Mod})$ *where* $\mathcal{A}$ *is closed under suspensions. Then the embedding functor* $K(\mathcal{A}) \to K(R\text{-Mod})$ *has a right adjoint.*

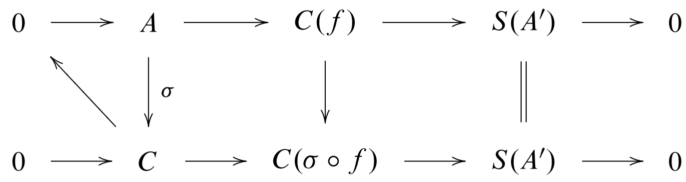
Proof. Our proof will be based on Proposition 5.1.2. So given an object $C \in K(R\text{-Mod})$ we need to find an $A \in K(\mathcal{A})$ (so $A \in \mathcal{A}$) and a morphism $A \to C$ in $K(R\text{-Mod})$ that has the desired universal property. Since $(\mathcal{A}, \mathcal{B})$ is complete, we know that for $C \in C(R\text{-Mod})$ we have an exact $0 \to B \to A \xrightarrow{\sigma} C \to 0$ in $C(R\text{-Mod})$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Our candidate for the morphism $A \to C$ in $K(R\text{-Mod})$ will be $[\sigma]$. So we need to argue that if $A' \in K(\mathcal{A})$ and if $[\sigma'] : A' \to C$ is any morphism in $K(R\text{-Mod})$, then there is a unique morphism $[f] : A' \to A$ in $K(R\text{-Mod})$ such that $[\sigma] \circ [f] \circ [\sigma']$. The existence of $[f]$ follows from the fact that $\sigma : A \to C$ is an $\mathcal{A}$-precover in $C(R\text{-Mod})$. For this means that given $\sigma : A' \to C$ in $C(R\text{-Mod})$ with $A' \in \mathcal{A}$, there is a morphism $f : A' \to A$ so that $\sigma \circ f = \sigma'$. But then $[\sigma] \circ [f] = [\sigma \circ f] = [\sigma']$.

For uniqueness we need to argue that if $g : A' \to A$ is any other morphism in $C(R\text{-Mod})$ such that $[\sigma] \circ [g] = [\sigma']$, then $[f] = [g]$, i.e. that $f \cong g$ (f is homotopic to g, see Definition 3.1.2). So finally we see that we need to prove that if $f, g : A' \to A$ are morphisms in $C(R\text{-Mod})$ such that $\sigma \circ f \cong \sigma \circ g$, then $f \cong g$.

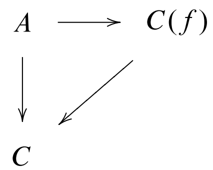
Since $\sigma \circ f \cong \sigma \circ g$ gives $\sigma \circ (f - g) \cong 0$, we see that we need to prove that $\sigma \circ (f - g) \cong 0$ implies $f - g \equiv 0$.

So writing f for $f - g$, we want to argue that $\sigma \circ g \cong 0$ implies $f \cong 0$.

By Proposition 3.3.2 this means that we need to prove that if $0 \to C \to C(\sigma \circ f) \to S(A') \to 0$ is split exact, then $0 \to A \to C(f) \to S(A') \to 0$ is split exact. But as noted in Chapter 4, we have the commutative diagram

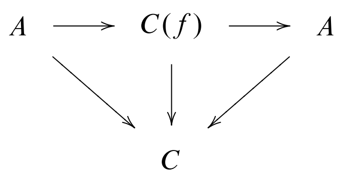


Since by hypothesis $\sigma \circ f \cong 0$, by Proposition 3.3.2 the bottom short exact sequence is split exact. Hence $C \to C(\sigma \circ f)$ admits a retraction $C(\sigma \circ f) \to C$. Then using the composition $C(f) \to C(\sigma \circ f) \to C$ we get a commutative diagram

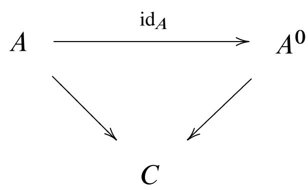


Since $\mathcal{A}$ is closed under extensions and suspensions, the exact sequence $0 \to A \to C(f) \to S(A') \to 0$ gives that $C(f) \in \mathcal{A}$. But $A \to C$ is an $\mathcal{A}$-precover of C. So the map $C(f) \to C$ has a lifting $C(f) \to A$.

We now want to apply Proposition 3.3.3. That result says that to get that $0 \to A \to C(f) \to S(A') \to 0$ is split exact (and so that $f \cong 0$) we only need that $A \to C(f)$ admits a retraction $C(f) \to A$ in $K(R\text{-Mod})$. So we argue that our lifting $C(f) \to A$ provides a retraction in $K(R\text{-Mod})$. This means that we need to argue that the composition $A \to C(f) \to A$ is homotopic to id_A. If we call this composition h then we need to argue that $\mathrm{id}_A - h \cong 0$. Since both diagrams



and



are commutative we see that $(A \to C) \circ (\mathrm{id}_A - h) = 0$. So this means that $\mathrm{Im}(\mathrm{id}_A - h) \subset \mathrm{Ker}(A \to C) = B$. So we can think of $\mathrm{id}_A - h$ as a morphism $k : A \to B$. Since $S(A) \in \mathcal{A}$ we have $\mathrm{Ext}^1(S(A), B) = 0$. So $0 \to B \to C(k) \to S(A) \to 0$ is split exact and $K = \mathrm{id}_A - h$ (as a morphism into B) is homotopic to 0. So then easily it is homotopic to 0 as a morphism into C. This completes our proof. $\square$

The proof of the next result follows the same pattern.

Theorem 5.1.8. *Let* $(\mathcal{A}, \mathcal{B})$ *be a complete cotorsion pair in* $C(R\text{-Mod})$ *where* $S^{-1}\mathcal{B} \subset \mathcal{B}$. *Then the embedding functor* $K(\mathcal{B}) \to K(R\text{-Mod})$ *has a left adjoint.*

Proof. We will give the main steps of the proof. Since $(\mathcal{A}, \mathcal{B})$ is complete, for $C \in C(R\text{-Mod})$ we have an exact sequence $0 \to C \overset{\tau}{\to} B \to A \to B$ with $B \in \mathcal{B}$ and $A \in \mathcal{A}$. Then $C \to B$ is a $\mathcal{B}$-preenvelope of C. We now want to argue that if $\tau' : C \to B'$ is a morphism in $C(R\text{-Mod})$ with $B' \in \mathcal{B}$, then there is a unique $[g] : B \to B'$ in $K(R\text{-Mod})$ with $[g] \circ [\tau] = [\tau']$. It suffices to argue that if $g \circ \tau \cong 0$, then $g \cong 0$. So we assume $g \circ \tau \equiv 0$. We have the commutative

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B' & \longrightarrow & C(g \circ \tau) & \longrightarrow & S(C) & \longrightarrow & 0 \\
 & & \Vert & & \downarrow & & \downarrow{\scriptstyle S(\tau)} & & \\
0 & \longrightarrow & B' & \longrightarrow & C(g) & \longrightarrow & S(B) & \longrightarrow & 0
\end{array}$$

with exact rows. By hypothesis $g \circ \tau \cong 0$ and so the top row is split exact and we have a section $S(C) \to C(g \circ \tau)$. But then we get the composition $S(C) \to C(g \circ \tau) \to C(g)$. Applying S^{-1}, we get $C \to S^{-1}(C(g \circ \tau)) \to S^{-1}(C(g))$. But $0 \to S^{-1}(B') \to S^{-1}(C(g)) \to B \to 0$ is exact and $B, S^{-1}(B') \in \mathcal{B}$. So $S^{-1}(C(g)) \in \mathcal{B}$. But since $\tau : C \to B$ is a $\mathcal{B}$-preenvelope, we get a morphism $B \to S^{-1}(C(g))$ that gives us a commutative diagram

$$\begin{array}{ccc}
C & & \\
\downarrow & \searrow & \\
B & \longrightarrow & S^{-1}(C(g))
\end{array}$$

The morphism $B \to S^{-1}(C(g))$ then gives a morphism $S(B) \to S(S^{-1}(C(g)) = C(g)$. To complete the proof, we need to show that $S(B) \to C(g)$ gives a section for $C(g) \to S(B)$ in $K(R\text{-Mod})$. An argument dual to the argument in the proof of Theorem 5.1.7 will give this fact and will complete the proof. □

To apply the results of this chapter we need a supply of complete cotorsion pairs $(\mathcal{A}, \mathcal{B})$ in $C(R\text{-Mod})$ with $\mathcal{A}$ closed under suspensions. The results of Chapter 4 provide some examples. In Chapter 7 we will see other ways of getting examples.

5.2 Exercises

1. Let $S : \mathcal{C} \to \mathcal{D}$ and $T : \mathcal{D} \to \mathcal{E}$ be functors and $\mu(X, Y) : \mathrm{Hom}(S(X), Y) \to \mathrm{Hom}(X, T(Y))$ be a natural bijection (i.e. μ: $\mathrm{Hom}(S(-), -) \to \mathrm{Hom}(-, T(-))$ is a natural isomorphism of functors). Taking X to be $T(Y)$ for $Y \in \mathcal{D}$ we have $\mathrm{id}_Y \in \mathrm{Hom}(T(Y), T(Y))$. Using $\mu(T(Y), Y)$ we get a morphism $\sigma : S(T(Y)) \to Y$. Argue that σ has the universal property described in Proposition 5.1.2.
2. Let $S_1 : \mathcal{C} \to \mathcal{D}$ and $S_2 : \mathcal{D} \to \mathcal{E}$ be functors having right adjoints $T_1 : \mathcal{D} \to \mathcal{C}$ and $T_2 : \mathcal{E} \to \mathcal{D}$, respectively. Argue that $T_1 \circ T_2$ is a right adjoint of $S_2 \circ S_1$.
3. Consider the category $\mathcal{C}$ whose objects are diagrams $f : M \to N$ of left R-modules and where a morphism of $M \to N$ to $M' \to N'$ is given by a commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & N \\ \downarrow & & \downarrow \\ M' & \longrightarrow & N' \end{array}$$

   Consider the functor $S : \mathcal{C} \to R\text{-Mod}$ where $S(M \to N) = M$. Find a left adjoint and a right adjoint of S.
4. Consider the forgetful functor T:Groups $\to$ Sets. Argue that T has a left adjoint (Hint: Recall the notion of a free group on a set.), but that G has no right adjoint. Suggestions: Consider the set $X = \{a, b\}$ with $a \neq b$. Argue that there is no group G and function $\tau : G \to X$ that has the desired universal property.

Chapter 6
Model Structures

Quillen, in his Homotopical Algebra, defined a model structure on a category. Hovey showed that there is a close connection between certain model structures on an abelian category and cotorsion pairs in that category. We will exhibit this close connection in the category $C(R\text{-Mod})$. Then we will use this connection to give examples of model structures on $C(R\text{-Mod})$.

6.1 Model Structures on $C(R\text{-Mod})$

Suppose we have two morphisms $i : A \to B$ and $p : X \to Y$ in $C(R\text{-Mod})$. By a morphism from i to p we just mean that we are given a commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

Definition 6.1.1. We say that i has the *left lifting property* with respect to p, or that p has the *right lifting property* with respect to i, if for every such commutative square we can complete the diagram to a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow{\scriptstyle i} & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

Example 6.1.2. In $C(R\text{-Mod})$, a morphism $i : 0 \to B$ has the left lifting property with respect to every epimorphism $p : X \to Y$ if and only if B is a projective complex. Similarly $X \to 0$ will have the right lifting property with respect to every momorphism $i : A \to B$ if and only if X is an injective complex.

Definition 6.1.3. By a *model structure* on $C(R\text{-Mod})$, we will mean that we have two classes of morphisms $\bar{\mathcal{C}}$ and $\bar{\mathcal{F}}$ called the cofibrations and the fibrations of the model structure. A morphism $i : A \to B$ in $\bar{\mathcal{C}}$ will be called a *trivial cofibration* if i is a homology isomorphism. Similarly $p : X \to Y$ in $\bar{\mathcal{F}}$ will be called a *trivial fibration* if p is a homology isomorphsm. Then the following conditions should be satisfied. A morphism $i : A \to B$ is a cofibration if and only if it has the left lifting property with respect to every trivial fibration, and $i : A \to B$ is a trivial cofibration if and only if it has the left lifting property with respect to every fibration. And a morphism $p : X \to Y$ is a fibration if and only if it has the right lifting property with respect to every trivial cofibration and is a trival fibration if and only if it has the right lifting property with respect to every cofibration. Finally we require that every morphism $f : C \to D$ in $C(R\text{-Mod})$ can be factored both as $f = p \circ i$ where i is a trivial cofibration and p is a fibration and as $f = p' \circ i'$ where i' is a cofibration and p' is a trivial fibration.

For later use and to motivate a later definition we prove the next result.

Lemma 6.1.4. *If a morphism $i : A \to B$ in $C(R\text{-Mod})$ has the left lifting property with respect to a morphism $p : X \to Y$ in $C(R\text{-Mod})$, then $0 \to \mathrm{Coker}(A \to B)$ has the left lifting property with respect to p.*

Proof. Let $C = \mathrm{Coker}(A \to B)$, then a commutative diagram

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \big\downarrow & & \big\downarrow p \\ 0 & \longrightarrow & Y \end{array}$$

gives rise to a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & 0 & \longrightarrow & X \\ i \big\downarrow & & \big\downarrow & & \big\downarrow p \\ B & \longrightarrow & C & \longrightarrow & Y \end{array}$$

If the diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \big\downarrow & \overset{h}{\nearrow} & \big\downarrow p \\ B & \longrightarrow & Y \end{array}$$

can be completed to a commutative diagram, then $A \to B \xrightarrow{h} X$ is the 0 morphism and so have an induced morphism $C = \text{Coker}(A \to B) \to X$. But then

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ C & \longrightarrow & Y \end{array}$$

is also commutative. So the claims easily follow from these observations. □

Lemma 6.1.5. *If a morphism $p : X \to Y$ has the right lifting property with respect to a morphism $i : A \to B$, then* $\text{Ker}(X \to Y) \to 0$ *has the right lifting property with respect to i.*

Proof. The proof is dual to that of Lemma 6.1.4. □

Definition 6.1.6. If $(\bar{\mathcal{C}}, \bar{\mathcal{F}})$ is a model structure on $C(R\text{-Mod})$, then an object $C \in C(R\text{-Mod})$ is said to be (*trivially*) *cofibrant* if $0 \to C$ is a (trivial) cofibration. An object $F \in C(R\text{-Mod})$ is said to be a (*trivially*) *fibrant* if $F \to 0$ is a (trivial) fibration.

Note that $0 \to C$ being a trivial cofibration just means that $0 \to C$ is a cofibration and that C is exact. Similary $F \to 0$ is a trivial fibration if it is a fibration and F is exact. So if we let $\mathcal{C}$ denote the class of cofibrant objects and if $\mathcal{E}$ is the class of exact complexes then $\mathcal{C} \cap \mathcal{E}$ is the class of trivially cofibrant ojects. Similarly $\mathcal{F} \cap \mathcal{E}$ will be the class of trivially fibrant objects if $\mathcal{F}$ is the class of fibrant objects.

Definition 6.1.7. Let $(\bar{\mathcal{C}}, \bar{\mathcal{F}})$ be a model structure on $C(R\text{-Mod})$ and let $\mathcal{C}$ and $\mathcal{F}$ be the cofibrant and fibrant objects respectively. Then $(\bar{C}, \bar{\mathcal{F}})$ is said to be a *special model structure* on $C(R\text{-Mod})$ if a morphism $i : A \to B$ is a (trivial) cofibration if and only if i is a monomorphism with (trivially) cofibrant cokernel, and if $p : X \to Y$ is a (trivial) fibration if and only if P is an epimorphism with (trivially) fibrant kernel.

We now exhibit the connection between model structures on $C(R\text{-Mod})$ and cotorsion pairs in $C(R\text{-Mod})$.

Theorem 6.1.8. *Let $(\bar{\mathcal{C}}, \bar{\mathcal{F}})$ be a special model structure on* $C(R\text{-Mod})$ *and let $\mathcal{C}$ and $\mathcal{F}$ be the cofibrant and fibrant objects for this model structure. Then $(\mathcal{C} \cap \mathcal{E}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{E})$ are complete cotorsion pairs in* $C(R\text{-Mod})$.

Proof. We will prove that $(\mathcal{C} \cap \mathcal{E}, \mathcal{F})$ is a complete cotorsion pair in $C(R\text{-Mod})$. The proof that $(\mathcal{C}, \mathcal{F} \cap \mathcal{E})$ is such a pair is similar.

Let $C \in \mathcal{C}$ and $F \in \mathcal{F} \cap \mathcal{E}$. We first prove that $\operatorname{Ext}^1(C, F) = 0$. Let $0 \to F \to U \to C \to 0$ be exact. This gives rise to the commutative square

$$\begin{array}{ccc} 0 & \longrightarrow & U \\ \big\downarrow & \nearrow & \big\downarrow \\ C & = & C \end{array}$$

Then since $0 \to C$ is a cofibration and since $U \to C$ is a special fibration the diagram can be completed to a commutative diagram. But then $C \to U$ is a section for $U \to C$ and so $0 \to F \to U \to C \to 0$ is split exact. Hence $\operatorname{Ext}^1(C, F) = 0$. This gives that $\mathcal{C} \subset {}^{\perp}(\mathcal{F} \cap \mathcal{E})$ and that

$$\mathcal{F} \cap \mathcal{E} \subset \mathcal{C}^{\perp}.$$

We now prove that ${}^{\perp}(\mathcal{F} \cap \mathcal{E}) \subset \mathcal{C}$. So suppose that $C \in {}^{\perp}(\mathcal{F} \cap \mathcal{E})$. That is, $\operatorname{Ext}^1(C, F) = 0$ for all $F \in {}^{\perp}(\mathcal{F} \cap \mathcal{E})$. We want to prove that $C \in \mathcal{C}$, i.e. that C is cofibrant. But this means that we need to prove that a commutative square

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow \\ C & \longrightarrow & Y \end{array}$$

with $X \to Y$ a trivial fibration can be completed to a commutatie diagram. Since our model structure is special, $X \to Y$ is an epimorphism with a trivially fibrant kernel. So let $0 \to F \to X \to Y \to 0$ be exact with $F \in \mathcal{F} \cap \mathcal{E}$. Then we have the exact sequence

$$\operatorname{Hom}(C, X) \to \operatorname{Hom}(C, Y) \to \operatorname{Ext}^1(C, F).$$

By hypothesis $\operatorname{Ext}^1(C, F) = 0$, so $\operatorname{Hom}(C, X) \to \operatorname{Hom}(C, Y) \to 0$ is exact. Then this in turn gives that

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow \\ C & \longrightarrow & Y \end{array}$$

can be completed to a commutative diagram. So this gives that $C \in \mathcal{C}$ and so that $\mathcal{C} = {}^{\perp}(\mathcal{F} \cap \mathcal{E})$.

We now argue that $\mathcal{C}^{\perp} \subset \mathcal{F} \cap \mathcal{E}$. So let $F \in \mathcal{C}^{\perp}$. To get that $F \in \mathcal{F} \cap \mathcal{E}$, we need that the commutative square

$$\begin{array}{ccc} A & \longrightarrow & F \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & 0 \end{array}$$

can be completed to a commutative diagram whenever $A \to B$ is a monomorphism with cofibrant cokernel. Let $0 \to A \to B \to C \to 0$ be exact with C cofibrant, i.e. $C \in \mathcal{C}$. Then the exact $\operatorname{Hom}(B, F) \to \operatorname{Hom}(A, F) \to \operatorname{Ext}^1(C, F) = 0$ shows that the diagram can be completed to a commutative diagram. So $\mathcal{C}^{\perp} \subset \mathcal{F} \cap \mathcal{E}$ and hence $\mathcal{C}^{\perp} = \mathcal{F} \cap \mathcal{E}$. So we have that $(\mathcal{C}, \mathcal{F} \cap \mathcal{E})$ is a cotorsion pair.

It only remains to show that our cotorsion pair is complete. Let $X \in C(R\text{-Mod})$. Then $f = 0 : 0 \to X$ has a factorization $f = p \circ i$ where i is a cofibration and p is a trivial fibration. Then if $i = 0 : 0 \to C$ and $p : C \to X$ we have that C is cofibrant and so $C \in \mathcal{C}$. Since $p : C \to X$ is a trivial fibration, it is an epimorphism with trivially fibrant kernel F. So we have an exact sequence $0 \to F \to C \to X \to 0$ with $C \in \mathcal{C}$ and $F \in \mathcal{F} \cap \mathcal{C}$. Given $Y \in C(R\text{-Mod})$, we establish the existence of an exact sequence $0 \to Y \to F' \to C' \to 0$ with $F' \in \mathcal{F} \cap \mathcal{E}$ and $C' \in \mathcal{C}$ by writing $f = 0 : Y \to 0$ as $f = p' \circ i'$ with i' a cofibration and p' a trivial fibration.

So $(\mathcal{C}, \mathcal{F} \cap \mathcal{E})$ is a complete cotorsion pair. As noted earlier, the argument that $(\mathcal{C} \cap \mathcal{E}, \mathcal{F})$ is a complete cotorsion pair is similar. □

We now want to prove a converse of Theorem 6.1.8.

We first prove two lemmas.

Lemma 6.1.9. *If $p : X \to Y$ and $p' : Y \to \mathbb{Z}$ are epimorphisms in $C(R\text{-Mod})$, then there is an exact sequence*

$$0 \to \operatorname{Ker}(p) \to \operatorname{Ker}(p' \circ p) \to \operatorname{Ker}(p') \to 0$$

Proof. We have $\operatorname{Ker}(p) \subset \operatorname{Ker}(p' \circ p)$. This give the map $\operatorname{Ker}(p) \to \operatorname{Ker}(p' \circ p)$. The map $\operatorname{Ker}(p' \circ p) \to \operatorname{Ker}(p')$ is $x \mapsto p(x)$. Then it is easy to check that these maps give us the desired exact sequence. □

Lemma 6.1.10. *Let*

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array}$$

be a commutative square in $C(R\text{-Mod})$ where i is a monomorphism with cokernel C and where P is an epimorphism with kernel F. If $\operatorname{Ext}^1(C, F) = 0$, then the diagram can be completed to a commutative diagram.

Proof. We have the exact sequences $0 \to A \to B \to C \to 0$ and $0 \to F \to X \to Y \to 0$. These give us the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & \operatorname{Hom}(C,X) & \longrightarrow & \operatorname{Hom}(C,Y) & \longrightarrow & \operatorname{Ext}^1(C,F)=0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & \operatorname{Hom}(B,F) & \longrightarrow & \operatorname{Hom}(B,X) & \longrightarrow & \operatorname{Hom}(B,Y) & & \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle i*} & & \\
 & & \operatorname{Hom}(A,F) & \longrightarrow & \operatorname{Hom}(A,X) & \xrightarrow{p*} & \operatorname{Hom}(A,Y) & & \\
 & & \downarrow & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\
0 & \longrightarrow & \operatorname{Ext}^1(C,F) & \longrightarrow & \operatorname{Ext}^1(C,X) & \xrightarrow{p^*} & \operatorname{Ext}'(C,Y) & &
\end{array}$$

where i^*, p^* and ∂ (twice) have the obvious meanings. We now argue by diagram chasing. We have that $f \in \operatorname{Hom}(A, X)$ and $g \in \operatorname{Hom}(B, Y)$ have the same images in $\operatorname{Hom}(A, Y)$. So since $\partial(i^*(g)) = 0$, we get that $\partial(f) = 0$. So there is an $\bar{h} \in \operatorname{Hom}(B, X)$ such that $\operatorname{Hom}(B, X) \to \operatorname{Hom}(A, X)$ maps $\bar{h}$ to f, i.e. such that $\bar{h} \circ i = f$. Then referring to the diagram we see that $g = p \circ \bar{h}$ is the kernel of i^* and so is in the image of $\operatorname{Hom}(C, Y) \to \operatorname{Hom}(B, Y)$. But $\operatorname{Hom}(C, X) \to \operatorname{Hom}(C, Y) \to 0$ is exact. So we can find $k \in \operatorname{Hom}(C, X)$ so that $\operatorname{Hom}(C, X) \to \operatorname{Hom}(C, X) \to \operatorname{Hom}(B, Y)$ maps k to $g - p \circ \bar{h}$. If we now let $h = \bar{h} + k \circ j$ where $j : B \to C$, then we can check that k makes the original diagram commutative. □

Theorem 6.1.11. *Suppose $\mathcal{C}$ and $\mathcal{F}$ are classes of objects of $C(R\text{-Mod})$ such that $(\mathcal{C} \cap \mathcal{E}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{E})$ are complete cotorsion pairs in $C(R\text{-Mod})$. Then if $\bar{\mathcal{C}}$ is the class of monomorphisms $A \to B$ in $C(R\text{-Mod})$ whose cokerenel is in $\mathcal{C}$ and if $\bar{\mathcal{F}}$ is the class of epimorphisms $p : X \to Y$ in $C(R\text{-Mod})$ whose kernel is in $\mathcal{F}$, then $(\bar{\mathcal{C}}, \mathcal{F})$ is a special model structure on $C(R\text{-Mod})$.*

Proof. Suppose $i : A \to B$ and $p : X \to Y$ are in $\mathcal{C}$ and $\mathcal{F}$ respectively. Let $0 \to A \to B \to C \to 0$ and $0 \to F \to X \to Y \to 0$ be exact with $C \in \mathcal{C}$ and $F \in \mathcal{F}$. Now suppose moreover that $F \in \mathcal{E}$, i.e. that $F \in \mathcal{F} \cap \mathcal{E}$. Then since $(\mathcal{C}, \mathcal{F} \cap \mathcal{E})$ is a cotorsion pair, $\operatorname{Ext}^1(C, F) = 0$. Then by Lemma 6.1.10, any commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

can be completed to a commutative diagram. So using the language of model structures, we have that every cofibration has the left lifting property with respect to every trivial fibration. The same type argument gives that every trivial cofibration has the left lifting property with respect to every fibration.

Now let $i : A \to B$ be a morphism which has the left lifting property with respect to every trivial fibration $p : X \to Y$. Then we want to argue that $i : A \to B$ is a monomorphism and that its cokernel is in $\mathcal{C}$.

We first argue that it is a monomorphism. Let I be any injective complex. Then since $(\mathcal{C}, \mathcal{F})$ is a cotorsion pair, $I \in \mathcal{F}$. In Chapter 1 we saw that every injective complex is exact. Therefore $I \in \mathcal{F} \cap \mathcal{E}$. So $p : I \to 0$ is a trival fibration. Now suppose that $A \subset I$. Then since $i : A \to B$ is a fibration, the square

$$\begin{array}{ccc} A & \longrightarrow & I \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow \\ B & \longrightarrow & 0 \end{array}$$

can be completed to a commutative diagram. But then $i : A \to B$ must be a monomorphism. So now we want to show that if $0 \to A \to B \to C \to 0$ is exact, then $C \in \mathcal{C}$. So if $F \in \mathcal{F} \cap \mathcal{C}$, we must show that $\mathrm{Ext}^1(C, F) = 0$.

So let $0 \to F \to I \to Y \to 0$ be exact with injective. Then $I \to Y \in \bar{\mathcal{F}}$. So $i : A \to B$ has the left lifting property with respect to $I \to Y$. Hence by Lemma 6.1.4, $0 \to C$ has the left lifting property with respect to $I \to Y$. But this means $\mathrm{Hom}(C, I) \to \mathrm{Hom}(C, Y) \to 0$ is exact. But this gives that $\mathrm{Ext}^1(C, F) = 0$.

So now we have that if $i : A \to B$ has the left lifting property with respect to every trivial $p \in \bar{\mathcal{F}}$, then $i \in \bar{\mathcal{C}}$. So since we have already shown that every $i \in \bar{\mathcal{C}}$ has the left lifting property with respect to every trivial $p \in \bar{\mathcal{F}}$.

The other claims about lifting properties can be proved in a similar or in a dual manner.

Then it now remains to prove that every morphism f has the factorizations $f = p \circ i = p' \circ i'$ as in the definition of a model structure.

We argue that f has a factorization $f = p \circ i$ with i a trivial cofibration and p a fibration. We will use the fact that $(\mathcal{C} \cap \mathcal{E}, \mathcal{F})$ is a complete cotorsion pair.

We first assume $f : A \to B$ is a monomorphism with cokernel L. Sine $(\mathcal{C} \cap \mathcal{E}, \mathcal{F})$ is complete, there is an exact sequence $0 \to F \to C \to L \to 0$ with $F \in \mathcal{F}$ and $C \in \mathcal{C} \cap \mathcal{E}$. If we form a pull back of

$$\begin{array}{ccc} & & C \\ & & \downarrow \\ B & \longrightarrow & L \end{array}$$

we get a commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & F & = & F & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & L & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

with exact rows and columns. So $i = (A \to B') \in \bar{\mathcal{C}}$ and i is a trivial cofibration since C is exact. And $p = (B' \to B) \in \mathcal{F}$. Since $f = p \circ i'$, we have the desired factorization.

Now we assume $f : X \to Y$ is an epimorphism with kernel K. Since $(\mathcal{C} \cap \mathcal{E}, \mathcal{F})$ is complete, we have an exact sequence $0 \to K \to F \to C \to 0$ with $F \in \mathcal{F}$ and $C \in \mathcal{C} \cap \mathcal{E}$. Now forming a pushout of

$$\begin{array}{ccc} K & \longrightarrow & X \\ \downarrow & & \\ F & & \end{array}$$

we get a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & K & \longrightarrow & X & \longrightarrow & Y & & \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & F & \longrightarrow & X' & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & C & = & C & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

with exact rows and columns. We see then that $(X' \to Y) \circ (X \to X')$ is the desired factorization.

Now we let $f : A \to B$ be any morphism. We can write f as the composition $A \xrightarrow{g} A \oplus B \xrightarrow{h} B$ with $A \to A \oplus B$ the map $x \mapsto (x, 0)$ and $A \oplus B \to B$ the map $(x, y) \mapsto f(x) + y$. Then since h is an epimorphism we have a factorization $h = p \circ i$ with i a trivial cofibration and p a fibration. But then i is a monomorphism and so $i \circ g$ is a monomorphism. So $i \circ g$ has a factorization $i \circ g = q \circ j$ with j a trivial cofibration and q a fibration.

So we get $f = h \circ g = p \circ i \circ g = (p \circ q) \circ j$ where j is a trivial cofibration. We have that both p and q are fibrations, so both are epimorphisms. Hence $p \circ q$ is an epimorphism. By Lemma 6.1.9 we have an exact sequece

$$0 \to \text{Ker}(q) \to \text{Ker}(p \circ g) \to \text{Ker}(p) \to 0.$$

But $\text{Ker}(q), \text{Ker}(p) \in \mathcal{F}$. So since $\mathcal{F}$ is closed under extensions, we get that $\text{Ker}(p \circ q) \in \mathcal{F}$. Hence $p \circ q \in \bar{\mathcal{F}}$, i.e. $p \circ q$ is a fibration. This gives us the desired decomposition of f into a trivial cofibration followed by a fibration. The argument for a decomposition into a cofibration followed by a trivial fibration is the same. □

Definition 6.1.12. If $\mathcal{A}$ and $\mathcal{B}$ are classes of objects of $C(R\text{-Mod})$ such that $(\mathcal{A} \cap \mathcal{E}, \mathcal{B})$ and $(\mathcal{A}, \mathcal{B} \cap \mathcal{E})$ are both complete cotorsion pairs in $C(R\text{-Mod})$, we say that $(\mathcal{A} \cap \mathcal{E}, \mathcal{B})$, $(\mathcal{A}, \mathcal{B} \cap \mathcal{E})$ is a *Hovey pair*. Theorems 6.1.11 and 6.1.8 say that there is a bijective correspondence between the collection of special model structures on $C(R\text{-Mod})$ and the collection of Hovey pairs in $C(R\text{-Mod})$.

Example 6.1.13. In section 4.3 we considered the Dold triplet $({}^{\perp}\mathcal{E}, \mathcal{E}, \mathcal{E}^{\perp})$. This triplet is hereditary (see Definition 4.4.3). By Theorem 4.4.4 ${}^{\perp}\mathcal{E} \cap \mathcal{E}$ consists of all the projective complexes. By then $({}^{\perp}\mathcal{E} \cap \mathcal{E}, R\text{-Mod})$ is trivially a complete cotorsion pair. But so is $({}^{\perp}\mathcal{E}, (R\text{-Mod}) \cap \mathcal{E}) = ({}^{\perp}\mathcal{E}, \mathcal{E})$. Hence we have a Hovey pair $({}^{\perp}\mathcal{E} \cap \mathcal{E}, R\text{-Mod}), ({}^{\perp}\mathcal{E}, \mathcal{E})$. Similarly we have a Hovey pair $(\mathcal{E}, \mathcal{E}^{\perp}), (R\text{-Mod}, \mathcal{E}^{\perp} \cap \mathcal{E})$. So this give us two special model structures on $C(R\text{-Mod})$.

6.2 Exercises

In the next set of exercises, let $(\bar{\mathcal{C}}, \bar{\mathcal{F}})$ be a special model structure on $C(R\text{-Mod})$ and let $\mathcal{C}$ and $\mathcal{F}$ be the classes of cofibration and fibrant objects for this model structure.

1. Argue that $\bar{\mathcal{C}}$ contains all isomorphisms and that it is closed under taking composition.
2. If $f : C \to D$ is a morphism in $C(R\text{-Mod})$ and if $C \xrightarrow{i} U \xrightarrow{p} D$ and $C \xrightarrow{i'} U' \xrightarrow{p'} D$ are two factorizations of f with i and i' trivial cofibrations and with p and p' fibrations, then argue that there is a morphism $h : U \to U'$ that makes the diagram

$$\begin{array}{ccccc} C & \longrightarrow & U & \longrightarrow & D \\ \| & & \downarrow & & \| \\ C & \longrightarrow & U' & \longrightarrow & D \end{array}$$

   commutative.
   Hint: Use Lemma 6.1.10.
3. Using the notation of problem 2, suppose that $C \to U \to D$ is such that the diagram

$$\begin{array}{ccccc} C & \longrightarrow & U & \longrightarrow & D \\ \| & & \downarrow & & \| \\ C & \longrightarrow & U & \longrightarrow & D \end{array}$$

   can only be completed by automorphisms $k : U \to U$ of U. In this case all $f = p \circ i$ a minimal factorization of f into a trivial cofibration followed by a fibration. Then:

a) Show that if $f = p \circ i$ and $f = p' \circ i'$ are both minimal factorizations of f then any $h : U \to U'$ that makes the diagram of Exercise 2 above commutative is an isomorphism.

b) If $C \to U \to D$ and $\bar{C} \to \bar{U} \to \bar{D}$ are minimal factorizations of $f : C \to D$ and $\bar{f} : \bar{C} \to \bar{D}$, argue that $C \oplus \bar{C} \to U \oplus \bar{U} \to D \oplus \bar{D}$ is a minimal factorization of $f \oplus \bar{f} : C \oplus \bar{C} \to D \oplus \bar{D}$.

c) Argue the converse of b)

d) If $C \in \mathcal{C}$ and $F \in \mathcal{F}$ describe minimal factorizations of $C \to 0$ and $0 \to F$.

Chapter 7
Creating Cotorsion Pairs

In this chapter we will be concerned with ways of getting complete cotorsion pairs in $C(R\text{-Mod})$. One method for creating such pairs is by starting with a complete cotorsion pair in R-Mod and then using this pair to find related pairs in $C(R\text{-Mod})$.

7.1 Creating Cotorsion Pairs in $C(R\text{-Mod})$ in a Termwise Manner

Let $\mathcal{A}$ be a class of objects of R-Mod. We will often regard $\mathcal{A}$ as a full subcategory of R-Mod. Then $C(\mathcal{A})$ will denote the class of objects $A \in C(R\text{-Mod})$ such that $A_n \in \mathcal{A}$ for each $n \in \mathbb{Z}$. We will also think of $C(\mathcal{A})$ as the corresponding full subcategory of $C(R\text{-Mod})$.

Now let $(\mathcal{A}, \mathcal{B})$ be a complete cotorsion pair in R-Mod. The question we will ask is whether $C(\mathcal{A})$ and $C(\mathcal{B})$ are components of complete cotorsion pairs in $C(R\text{-Mod})$.

We first note that it is certainly not true in general that $(C(\mathcal{A}), C(B))$ is even a cotorsion pair in $C(R\text{-Mod})$. For example, let $(\mathcal{A}, \mathcal{B})$ be the pair $(R\text{-Mod}, \text{Inj})$ with $\text{Inj} \subset R\text{-Mod}$ the class of injective modules. Then $C(R\text{-Mod})^{\perp}$ is the class of injective complexes say I. We know that $I \in C(\text{Inj})$. But in Chapter 1 we saw that every injective complex is exact. But not every element of $C(\text{Inj})$ is exact (unless $R = 0$).

So our question will be whether $(C(\mathcal{A}), C(\mathcal{A})^{\perp})$ and $(C(\mathcal{B})^{\perp}, C(\mathcal{B}))$ form complete cotorsion pairs in $C(R\text{-Mod})$. Recall that a pair $(\mathcal{A}, \mathcal{B})$ in R-Mod is complete if it is cogenerated by a set (Volume I, Theorem 7.4.1). We now use this result to get the following.

Theorem 7.1.1. *If* $(\mathcal{A}, \mathcal{B})$ *is a cotorsion pair in* R-Mod *which is cogenerated by a set, then* $(C(\mathcal{B})^{\perp}, C(\mathcal{B}))$ *is a cotorsion pair in* $C(R\text{-Mod})$ *which is cogenerated by a set.*

Proof. We only need to find a set $\mathcal{T}$ of objects of $C(R\text{-Mod})$ such that $\mathcal{T}^{\perp} = C(\mathcal{B})$. Let $\mathcal{S}$ be a set of objects of R-Mod such that $\mathcal{S}^{\perp} = \mathcal{B}$. Now let $\mathcal{T} = \{S^k(\bar{M}) \in M \in \mathcal{S}, k \in \mathbb{Z}\}$. Then by Proposition 2.1.3, we get that $\mathcal{T}^{\perp} = C(\mathcal{B})$. This gives the result. □

An example of interest is when $\mathcal{B} = \text{Inj}$. It would be nice to have a description of $C(\text{Inj})^{\perp}$.

With the object of creating Hovey pairs (see Chapter 6), we prove the next result.

Corollary 7.1.2. $((C(\mathcal{B}) \cap \mathcal{E})^{\perp}, C(\mathcal{B}) \cap \mathcal{E})$ *is a cotorsion pair which is cogenerated by a set where* $\mathcal{E}$ *is the class of exact complexes.*

Proof. We let $\mathcal{U}$ be the set of objects of $C(R\text{-Mod})$ with $\mathcal{U} = \mathcal{T} \cup \mathcal{T}'$ where $\mathcal{T}' = \{S^k(\underline{R}) : k \in \mathbb{Z}\}$. Then $\mathcal{U}^{\perp} = \mathcal{T}^{\perp} \cap (\mathcal{T}')^{\perp}$. So by the above and by Corollary 2.1.7, we get that $\mathcal{U}^{\perp} = C(\mathcal{B}) \cap \mathcal{E}$. □

The proof of the analogous result for $(C(\mathcal{A}), C(\mathcal{A})^{\perp})$ and $(C(\mathcal{A}) \cap \mathcal{E}, (C(\mathcal{A}) \cap \mathcal{E})^{\perp})$ is not as direct. The tools we need will be provided by the so-called Hill lemma. We note that this lemma has proved to be of great use in the work of Trlifaj et al.

7.2 The Hill Lemma

The Hill lemma is a way of creating a plentiful supply of submodules of a module with a given filtration, but where these submodules have nice properties.

Throughout this section we will suppose that $\mathcal{C}$ is a set of left R-modules. We will also suppose that k is an infinite regular cardinal with $|R| < \kappa$ and with $|C| < \kappa$ for all $C \in \mathcal{C}$. We will also let M be a left R-module with a given $\mathcal{C}$-filtration $(M_\alpha \mid \alpha \leq \sigma)$.

We will use the following procedure to create a family of submodules of M. For each $\alpha + 1 \leq \sigma$, we have $|\frac{M_{\alpha+1}}{M_\alpha}| < \kappa$. So choosing a set X of representatives of the cosets in $\frac{|M_{\alpha+1}|}{M_\alpha}$, we let A_α be the submodule of $M_{\alpha+1}$ generated by X. Then since $|X| < \kappa$, $|R| < \kappa$, and κ is infinite, we get $|A_\alpha| < \kappa$. We also have $M_{\alpha+1} = M_\alpha + A_\alpha$. We do this for each $\alpha + 1 \leq \sigma$ and so have the fixed $A_\alpha \subset M$. By transfinite induction we get that $M_\beta = \sum_{\alpha<\beta} A_\alpha$ for each $\beta \leq \sigma$.

We note that if we begin with a module M and a family $(A_\alpha)_{\alpha<\sigma}$ of submodules for some ordinal σ such that $M = \sum_{\alpha<\sigma} M_\alpha$, then if we define $M_\beta = \sum_{\alpha<\beta} M_\alpha$ for each $\beta \leq \sigma$, then we have a filtration of M.

Now suppose $S \subset \{\alpha \mid \alpha < \sigma\}$. Then we define $M(S)$ to be $\sum_{\alpha \in S} M_\alpha$.

There are then two natural ways to get a filtration on $M(S)$. One is by taking the induced filtration on $M(S)$, i.e. the filtration $(M(S) \cap M_\beta \mid \beta \leq \sigma)$. Another way would be to define the β^{th} term of the filtration to be $\sum_{\alpha \in S, \alpha<\beta} M_\alpha$.

Definition 7.2.1. We say $S \subset \{\alpha \mid \alpha < \sigma\}$ is closed if the two filtrations above are the same, i.e. if

$$M(S) \cap M_\beta = \sum_{\alpha \in S, \alpha<\beta} A_\alpha$$

for all $\beta \leq \sigma$.

Eventually our Hill class of submodules will be the $M(S)$ where S is closed. But first we want to get other ways to guarantee that S is closed.

Noting that $\sum_{\alpha\in S,\alpha<\beta} A_\alpha \subset M(S)\cap M_\beta$ for any S and any $\beta\le\sigma$, we have that S is closed if and only if $M(S)\cap M_\beta \subset \sum_{\alpha\in S,\alpha<\beta} A_\alpha$ for every $\beta\le\sigma$.

We want other ways to guarantee that S is closed.

Lemma 7.2.2. *For $S\subset\{\alpha\mid\alpha<\sigma\}$, we have $M(S)\cap M_\beta\subset\sum_{\alpha\in S,\alpha<\beta} A_\alpha$ for all $\beta\le\sigma$ if and only if the containment holds for all $\beta\in S$.*

Proof. If the containment holds for all β, then it holds for $\beta\in S$. Conversely suppose it holds for all $\beta\in S$. We want to prove that it holds for all β.

If $\beta\le\sigma$ is such that $\alpha<\beta$ for all $\alpha\in S$ then

$$M(S)\cap M_\beta = M(S) = \sum_{\alpha\in S} A_\alpha = \sum_{\alpha\in S,\alpha<\beta} A_\alpha$$

So our containment holds. If β is such that $\beta\le\alpha$ for some $\alpha\in S$, let $\beta'\in S$ be the least element of S such that $\beta\le\beta'$. Then

$$M(S)\cap M_\beta\subset M(S)\cap M_{\beta'}\subset \sum_{\alpha\in S,\alpha<\beta'} A_\alpha = \sum_{\alpha\in S,\alpha<\beta} A_\alpha.$$

So the containment holds for β. □

The next result gives an even simpler check that $S\subset\{\alpha\mid\alpha<\sigma\}$ is closed.

Lemma 7.2.3. *S is closed if and only if for all $\beta\in S$,*

$$M_\beta\cap A_\beta\subset\sum_{\alpha\in S,\alpha<\beta} A_\alpha$$

Proof. The condition is obviously necessary. Now suppose it holds for all β. We then want to argue that $M(S)\cap M_\beta\subset\sum_{\alpha\in S,\alpha<\beta} A_\alpha$ for all $\beta\in S$. If this condition does not hold for all $\beta\in S$, we choose the least β for which it does not.

Then let $x\in M(S)\cap M_\beta$, $x\notin\sum_{\alpha\in S,\alpha<\beta} A_\alpha$. Then since $x\in M(S)=\sum_{\alpha\in S} A_\alpha$, we can write $x=x_1+\cdots+x_k$ with $x_1\in A_{\alpha_1},\ldots,x_k\in A_{\alpha_k}$, where $\alpha_1<\alpha_2<\cdots<\alpha_k$ and where $\alpha_1,\ldots,\alpha_k\in S$. Now we also suppose that α_k is the least element of S we can get by this procedure.

Then $x_k=x-(x_1+\cdots+x_{k-1})\in M_{\alpha_k}\cap A_{\alpha_k}\subset\sum_{\alpha\in S,\alpha<\alpha_k} A_\alpha$. So then $x=(x_1+\cdots+x_{\alpha_{k-1}})+x_k\in\sum_{\alpha\in S,\alpha<\alpha_k} A_\alpha$. So writing x as a sum in $\sum_{\alpha\in S,\alpha<\alpha_k} A_\alpha$, we contradict the choice of α_k. □

Lemma 7.2.4. *Let $(S_i)_{i\in I}$ be any family of closed subsets of $\{\alpha\mid\alpha\le\sigma\}$. Then $S=\bigcup_{i\in I} S_i$ is also closed.*

Proof. This follows easily from the previous lemma. □

Note that $M(S) = M(\bigcup_{i\in I} S_i) = \sum_{i\in I} M(S_i)$.

Lemma 7.2.5. *Let $S \subset \{\alpha \mid \alpha < \sigma\}$ be closed. If $\beta \le \sigma$ is such that $\alpha \le \beta$ for all $\alpha \in S$ and if $M_\beta \cap A_\beta \subset \sum_{\alpha\in S,\alpha<\beta} A_\alpha$, then $S \cup \{\beta\}$ is closed.*

Proof. Again, we only need to use Lemma 7.2.3 above. □

We want to use this lemma to guarantee a plentiful supply of closed S.

Lemma 7.2.6. *If $\alpha \le \sigma$, then there is a closed subset S with $\alpha \in S$, $|S| < \kappa$ and with α the largest element of S.*

Proof. We proceed by transfinite induction. So suppose the claim holds for all $\alpha < \beta$ for some $\beta \le \sigma$. We argue that it holds for β. We have

$$M_\beta \cap A_\beta \subset M_\beta = \sum_{\alpha<\beta} A_\alpha.$$

Since $|A_\beta| < \kappa$, we have $|M_\beta \cap A_\beta| < \kappa$. So there is a set T of α with $\alpha < \beta$ with $|T| < \kappa$ and with $M_\beta \cap A_\beta \subset \sum_{\alpha\in T} A_\alpha$.

For each $\alpha \in T$ we have $\alpha < \beta$. So by our induction hypothesis, we have a closed subset S_α with $|S_\alpha| < \kappa$ and with the α the largest element of S_α. But then $S' = \bigcup_{\alpha\in T} S_\alpha$ is closed by Lemma 7.2.4. Also $|S'| \le \sum_{\alpha\in T} |S_\alpha| < \kappa$ since κ is an infinite regular cardinal. Now let $S = S' \cup \{\beta\}$. Then using Lemma 7.2.5, we have that S is closed. □

Corollary 7.2.7. *If $S \subset \{\alpha|\alpha < \sigma\}$ is closed and $\beta \le \sigma$, then there is a closed T with $S \subset T$, $\beta \in T$ and with $|T - S| < \kappa$.*

Proof. Apply the previous lemma and Lemma 7.2.4. □

Lemma 7.2.8. *If $S \subset T$ are closed subsets of $\{\alpha|\alpha \le \sigma\}$, then $M(T)/M(S) \in$* Filt($\mathcal{C}$).

Proof. We argue that $M(T) \in \text{Filt}(\mathcal{C})$. For $\beta + 1 \le \sigma$, we have

$$M(T) \cap M_{\beta+1}/(M(T) \cap M_\beta) = \sum_{\alpha\in T,\alpha<\beta+1} A_\alpha \Big/ \sum_{\alpha\in T,\alpha<\beta} A_\alpha$$

If $\beta \notin T$, we see that this quotient is 0. If $\beta \in T$, the quotient is isomorphic to

$$A_\beta/\Big(A_\beta \cap \sum_{\alpha\in T,\alpha<\beta} S_\alpha\Big) = A_\beta/(A_\beta \cap M_\beta) \cong \frac{A_\beta + M_\beta}{M_\beta} = \frac{M_{\beta+1}}{M_\beta}$$

So this gives that the filtration on $M(T)$ induced from that on M is a $\mathcal{C}$-filtration. So $M(T) \in \text{Filt}(\mathcal{C})$.

The argument that $M(T)/M(S) \in \text{Filt}(\mathcal{C})$ is similar. □

Definition 7.2.9. With all our hypotheses in place, we call the *Hill class* of submodules of M the submodules $M(S)$ for $S \subset \{\alpha \mid \alpha < \sigma\}$ closed.

This class will be denoted $\mathcal{H}(M)$. We note that it depends on the filtration of M, on the choice of κ and on the choice of the A_α. We will call $\mathcal{H}(M)$ a Hill class of submodules relative to κ. We now record all that we have proved about $\mathcal{H}(M)$. Note that $0 \in \mathcal{H}(M)$ since $S = \phi$ is closed. Similarly $M \in \mathcal{H}(M)$.

Theorem 7.2.10. a) *If $N \subset P$ with $N, P \in \mathcal{H}(M)$ then $P/N \in$ Filt($\mathcal{C}$) (and so $N \in$ Filt($\mathcal{C}$) and $M/N \in$ Filt($\mathcal{C}$)).*

b) *$\mathcal{H}(M)$ is closed under sums.*

c) *If $N \in \mathcal{H}(M)$ and $X \subset M$ with $|X| < \kappa$, then there is a $P \in \mathcal{H}(M)$ with $N \subset P$, $X \subset P$ and $|P/N| < \kappa$.*

Proof. The proofs of these claims are in the above. The claim for c) is just Corollary 7.2.7. For b) we recall that $M(\bigcup_{i \in I} S_i) = \sum_{i \in I} M(S_i)$ when $(S_i)_{i \in I}$ is a family of closed sets and that $\bigcup_{i \in I} S_i$ is closed (Lemma 7.2.4). □

We now begin with $M \in$ Filt($\mathcal{C}$). Suppose we have a direct sum decomposition $M = M_1 \oplus M_2$. It is not necessarily true that $M_1, M_2 \in$ Filt($\mathcal{C}$). But using Hill classes we can prove a related result.

We first give some terminology. For a submodule $N \subset M$ we say N is *homogeneous* with respect to the direct sum decomposition $M = M_1 \oplus M_2$ if $N = (M_1 \cap N) \oplus (M_2 \oplus N)$. Note that sum of and intersection of homogeneous submodules is homogeneous.

Proposition 7.2.11. *If $X \subset M$ is a subset with $|X| < \kappa$, then there is a homogeneous Q with $Q \in \mathcal{H}(M)$, $X \subset Q$ and with $|Q| < \kappa$.*

Proof. By c) of Theorem 7.2.10 above there is a $P \in \mathcal{H}(M)$ with $X \subset P$ and $|P| < \kappa$. Of course, P may not be homogeneous. But $P \subset P_1 \oplus P_2$ where P_1 and P_2 are the projections of P onto M_1 and M_2 (using the decomposition $M = M_1 \oplus M_2$). Also $|P_1| < \kappa$, $|P_2| < \kappa$ and $|P_1 \oplus P_2| < \kappa$. So we can find $P' \in \mathcal{H}(M)$ with $P_1 \oplus P_2 \subset P'$ and with $|P'| < K$. Continuing in this manner, we find $P \subset P' \subset P'' \subset \cdots$ with $P^{(k)} \subset \mathcal{H}(M)$, $P^{(k)}| < \kappa, (P^{(k)})_1 \oplus (P^{(k)})_2 \subset P^{(k+1)}$. Then if $Q = \bigcup_{k=0}^{\infty} P^{(k)}$, we quickly check that Q satisfies the desired properties. □

Corollary 7.2.12. *If $M = M_1 \oplus M_2$, then M has a filtration $(M^*_\alpha \mid \alpha \leq \tau)$ with each $M^*_\alpha \in \mathcal{H}(M)$ where each M^*_α is homogeneous and $|M^*_{\alpha+1}/M^*_\alpha| < \kappa$ whenever $\alpha + 1 \leq \tau$.*

Proof. This follows from the previous result. □

Now we note that since each M_α^* is homogeneous with respect to the direct sum decomposition $M = M_1 \oplus M_2$, each $M_{\alpha+1}^*/M_\alpha^*$ has a natural direct sum decomposition. But also recall that we have $M_{\alpha+1}^*/M_\alpha^* \in \mathrm{Filt}(\mathcal{C})$. From this we immediately get the next result

Corollary 7.2.13. *M_1 has a filtration by modules which are direct summands of modules in* Filt($\mathcal{C}$) *and which have cardinality less than $\mathcal{K}$.*

So we let $\mathcal{D}$ be a set of representatives of all modules D which are direct summands of modules in Filt($\mathcal{C}$) but where $|D| < \kappa$. So with this terminology, we have that M_1 admits a $\mathcal{D}$-filtration.

We will apply what we have done to a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in R-Mod which is cogenerated by a set $\mathcal{C}$. For convenience we can suppose $R \in \mathcal{C}$. We then let κ be an infinite regular cardinal so that $|C| < \kappa$ for all $C \in \mathcal{C}$. Then we know that every $M \in \mathcal{A}$ is a direct summand of a module having a $\mathcal{C}$-filtration. Using the $\mathcal{D}$ above, we get that every $M \in \mathcal{A}$ has a $\mathcal{D}$-filtration. Note that by our choice of $\mathcal{D}$, we have $\mathcal{D} \subset \mathcal{A}$.

In fact we can let $\mathcal{D}$ be a set of representatives of all $D \in \mathcal{A}$ such that $|D| < \kappa$. Then with this $\mathcal{D}$ we get that $\mathcal{D}$ cogenerates $(\mathcal{A}, \mathcal{B})$, $R \in \mathcal{D}$, and every $M \in \mathcal{A}$ has a $\mathcal{D}$-filtration. We also note that $\mathcal{D}$ has some obvious closure properties. First, any direct summand of an element of $\mathcal{D}$ is isomorphic to an element of $\mathcal{D}$. Then if M has $\mathcal{D}$-filtration $(M_\alpha | \alpha \le \sigma)$ where $\sigma < \kappa$, then $M \in \mathcal{A}$ and $|M| < \kappa$. So M is then isomorphic to an element of $\mathcal{D}$. Thus we can now prove our desired result.

Theorem 7.2.14. *If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in R-*Mod *which is cogenerated by a set, then $(C(\mathcal{A}), C(\mathcal{A})^\perp)$ is a cotorsion pair in $C(R$-*Mod*$)$ which is cogenerated by a set.*

Proof. We prove that $C(\mathcal{A})$ satisfies the conditions of Theorem 4.1.13. Since $\mathrm{Filt}(\mathcal{A}) = \mathcal{A}$, clearly $\mathrm{Filt}(C(\mathcal{A})) = C(\mathcal{A})$. So a) of that Theorem is satisfied. Similarly $C(\mathcal{A})$ satisfies b). Since $R \in \mathcal{A}$, we get $C(\mathcal{A})$ satisfies d). So we now prove $C(\mathcal{A})$ satisfies condition c). We let $\mathcal{D} \subset \mathcal{A}$ be as in the above. So $\mathcal{D}$ is a set which cogenerates $(\mathcal{A}, \mathcal{B})$ and $|D| < \kappa$ for all $D \in \mathcal{D}$ where κ is an infinite regular cardinal. We also have the closure properties for $\mathcal{D}$ given above. We want to prove that $C(\mathcal{A})$ is κ-deconstructible. This means that every $A \in C(\mathcal{A})$ has a filtration $(A^\alpha | \alpha \le \sigma)$ with $|A^{\alpha+1}/A^\alpha| \le \kappa$ and with $A^{\alpha+1}/A^\alpha \in \mathcal{A}$ for every $\alpha + 1 \le \sigma$.

To get the term $A_{n_0+1}^{\alpha+1}$, we consider a set $X \subset A_{n_0}^{\alpha+1}$ of representatives of the cosets of $A_{n_0}^{\alpha+1}/A_{n_0}^\alpha$. Then $d_{n_0}(X) \subset A_{n_0+1}$. We have $A_{n_0+1}^\alpha \in \mathcal{H}(A_{n_0+1})$ and $|d_{n_0}(X)| < \kappa$. So we can find $A_{n_0+1}^{\alpha+1} \in \mathcal{H}(A_{n_0+1})$ such that $A_{n_0+1}^\alpha \subset A_{n_0+1}^{\alpha+1}$, $d_{n_0}(X) \subset A_{n_0+1}^{\alpha+1}$ and such that $|A_{n_0+1}^{\alpha+1}/A_{n_0+1}^\alpha| < \kappa$. Note that by our choice of X, and since $d_{n_0}(A_{n_0}^\alpha) \subset A_{n_0+1}^\alpha$, we get $d_{n_0}(A_{n_0}^{\alpha+1}) \subset A_{n_0+1}^{\alpha+1}$. We continue this method and construct $A^{\alpha+1}_{n_0+2}, A^{\alpha+1}_{n_0+3}, \ldots$. Then letting $A^{\alpha+1}_n = A^{\alpha}_n$ if $n < n_0$, we have the desired complex $A^{\alpha+1} \subset A$. So now it easy to see that A has the desired filtration. □

Using the analogous method along with the zig-zag procedure of Remark 4.1.10, we get the proof of the next result.

Theorem 7.2.15. *If* $(\mathcal{A}, \mathcal{B})$ *is a cotorsion pair in* R-Mod *which is cogenerated by a set, then* $(C(\mathcal{A}) \cap \mathcal{E}, (C(\mathcal{A}) \cap \mathcal{E})^{\perp})$ *is a cotorsion pair in* $C(R$-Mod$)$ *which is cogenerated by a set.*

7.3 More Cotorsion Pairs

In the last section we considered one way of using a cotorsion pair in R-Mod to create cotorsion pairs in $C(R$-Mod$)$. Starting with the pair $(\mathcal{A}, \mathcal{B})$ in R-Mod, we consider the classes $C(\mathcal{A})$ and $C(\mathcal{B})$ in $C(R$-Mod$)$. So we considered complexes $A \in C(\mathcal{A})$ each of whose terms are in $\mathcal{A}$ and $B \in C(\mathcal{B})$ each of whose terms are in $\mathcal{B}$.

So, for example, if $(\mathcal{A}, \mathcal{B}) = (\text{Proj}, R\text{-Mod})$ with Proj the class of projective left R-modules, we get the $P \in C(\text{Proj})$. These consist of all complexes P such that P_n is projective for each $n \in \mathbb{Z}$. But as we saw in Section 1.4, these are not the projective complexes. According to Theorem 1.4.7, $P \in C(R$-Mod$)$ is projective if and only if P is exact and each $Z_n(P)$ is a projective module. This observation suggests another way of creating cotorsion pairs in $C(R$-Mod$)$ from pairs in R-Mod.

Definition 7.3.1. If $\mathcal{A}$ is a class of objects of R-Mod, we let $\tilde{\mathcal{A}}$ be the class of $A \in C(R$-Mod$)$ such that A is exact and such that $Z_n(A) \in \mathcal{A}$ for each n.

Note that if $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in R-Mod, then if $A \in \tilde{\mathcal{A}}$ we have an exact sequence

$$0 \to Z_n(A) \to A_n \to Z_{n-1}(A) \to 0$$

for each n. So since $\mathcal{A}$ is closed under extensions, we get that $A_n \in \mathcal{A}$ for each n. So $\tilde{A} \subset C(\mathcal{A})$, and in fact $\tilde{A} \subset C(\mathcal{A}) \cap \mathcal{E}$ with $\mathcal{E}$ the class of exact sequences.

Our object now is to use $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ to create cotorsion pairs in $C(R$-Mod$)$.

Theorem 7.3.2. *If* $(\mathcal{A}, \mathcal{B})$ *is cogenerated by a set then* $(\tilde{\mathcal{A}}, \tilde{\mathcal{A}}^{\perp})$ *and* $({}^{\perp}\tilde{\mathcal{B}}, \tilde{\mathcal{B}})$ *are cotorsion pairs in* $C(R$-Mod$)$ *each of which are cogenerated by a set.*

Proof. The argument for $(\tilde{\mathcal{A}}, \tilde{\mathcal{A}}^{\perp})$ is now a familiar one. We use the zig-zag procedure of Remark 4.1.10 and the Hill classes of Section 7.2 (as in the proof of Theorem 7.2.14).

The argument that $({}^{\perp}\tilde{\mathcal{B}}, \tilde{\mathcal{B}})$ is a cotorsion pair which is cogenerated by a set is similar to the proof of Theorem 7.1.1. So we need to find a set $\mathcal{T}$ of complexes so that $\mathcal{T}^{\perp} = \tilde{\mathcal{B}}$. So let $\mathcal{S}$ be a set of modules that cogenerate $(\mathcal{A}, \mathcal{B})$. Then we let $\mathcal{T}$ consist of all $S^k(\underline{R})$, all $S^k(\bar{M})$ and all $S^k(\underline{M})$ where $k \in \mathbb{Z}$ and $M \in \mathcal{S}$.

We first argue that $\mathcal{T}^{\perp} \subset \tilde{B}$. Let $B \in \mathcal{T}^{\perp}$. Since $S^k(\underline{R}) \in \mathcal{T}$ for all k we get that B is exact by Corollary 2.1.7. By Proposition 2.1.4 and the fact that $S^k(\bar{M}) \in \mathcal{T}$ for all k we get that $B_n \in \mathcal{B} = \mathcal{S}^{\perp}$ for all n. We now want to prove that $Z_n(B) \in \mathcal{B}$ for all n.

We argue that $Z_1(B) \in \mathcal{B}$. The argument that $Z_n(B) \in \mathcal{B}$ for any n follows using suspensions.

Since B is exact, we have an exact sequence

$$0 \to Z_1(B) \to B_1 \to Z_0(B) \to 0.$$

Now let $M \in \mathcal{S}$. We want to show that $\mathrm{Ext}^1(M, Z_1(B)) = 0$. Since $B_1 \in \mathcal{B} = \mathcal{S}^{\perp}$, we do have that $\mathrm{Ext}^1(M, B_1) = 0$. So using the exact sequence

$$\mathrm{Hom}(M, B_1) \to \mathrm{Hom}(M, Z_0(B)) \to \mathrm{Ext}^1(M, Z_1(B)) \to \mathrm{Ext}^1(M, B_1) = 0$$

we see that we only need to argue that $\mathrm{Hom}(M, B_1) \to \mathrm{Hom}(M, Z_0(B)) \to 0$ is exact.

To do so consider $\underline{M}$ and a morphism $f : \underline{M} \to B$. Such a morphism is given by a linear map $M \to Z_0(B)$. Since $\mathrm{Ext}^1(S(\underline{M}), B) = 0$, we get that $\underline{M} \to B$ is homotopic to 0 by Proposition 3.3.2. So $f \cong 0$. But then $\mathcal{S}$ is (essentially) given by a linear map $M \to B_1$ that lifts $M \to Z_0(B)$. So we get that $\mathrm{Hom}(M, B_1) \to \mathrm{Hom}(M_1 Z_0(B)) \to 0$ is exact. So we have $Z_1(B) \in \mathcal{S}^{\perp} = \mathcal{B}$, and similarly that $Z_n(B) \in \mathcal{B}$ for all n. So $B \in \tilde{\mathcal{B}}$.

But conversely, if $B \in \tilde{\mathcal{B}}$, then reversing these arguments we get that $B \in \mathcal{T}^{\perp}$ and so that $\tilde{\mathcal{B}} = \mathcal{T}^{\perp}$. Hence $({}^{\perp}\tilde{\mathcal{B}}, \tilde{\mathcal{B}})$ is a cotorsion pair which is cogenerated by a set. □

We have the two cotorsion pairs $(\tilde{\mathcal{A}}, \tilde{\mathcal{A}}^{\perp})$, $({}^{\perp}\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ in $C(R\text{-Mod})$, each of which is cogenerated by sets. We want to prove that if the original cotorsion pair $(\mathcal{A}, \mathcal{B})$ is hereditary, then this pair is a Hovey pair (see Definition 6.1.12). Using that definition, to say $(\tilde{\mathcal{A}}, \tilde{\mathcal{A}}^{\perp})$, $({}^{\perp}\tilde{\mathcal{B}}, \tilde{\mathcal{B}})$ is a *Hovey pair* means that $\tilde{\mathcal{A}} = {}^{\perp}\tilde{\mathcal{B}} \cap \mathcal{E}$ and $\mathcal{E} \cap \tilde{\mathcal{A}}^{\perp} = \tilde{\mathcal{B}}$ where $\mathcal{E}$ is the class of exact sequences. So with this in mind we prove the next result.

Theorem 7.3.3. *If* $(\mathcal{A}, B)$ *is a hereditary cotorsion pair in* R-Mod*, then* $(\tilde{\mathcal{A}}, \tilde{\mathcal{A}}^{\perp})$*,* $({}^{\perp}\tilde{\mathcal{A}}, \tilde{\mathcal{A}})$ *is a Hovey pair in* $C(R\text{-Mod})$.

Proof. We argue that $\tilde{A} = {}^{\perp}\tilde{\mathcal{B}} \cap \mathcal{E}$. The proof that $\mathcal{E} \cap \tilde{\mathcal{A}}^{\perp} = \tilde{B}$ will then follow from a similar argument. We first argue that $\tilde{A} \subset {}^{\perp}\tilde{\mathcal{B}} \cap \mathcal{E}$. We have $\tilde{A} \subset \mathcal{E}$ by the definition on $\tilde{\mathcal{A}}$. To get $\tilde{\mathcal{A}} \subset {}^{\perp}\tilde{\mathcal{B}}$, we need to show that $\mathrm{Ext}^1(A, B) = 0$ for $A \in \tilde{\mathcal{A}}$ and $B \in \tilde{\mathcal{B}}$. We think of the d of B as a morphism $B \to S(Z(B))$. Since B is exact, this gives rise to an exact sequence

$$0 \to Z(B) \to B \to S(Z(B)) \to 0.$$

So to get $\mathrm{Ext}^1(A, B) = 0$ it suffices to show that

$$\mathrm{Ext}^1(A, Z(B)) = 0 \quad \text{and} \quad \mathrm{Ext}^1(A, S(Z(B))) = 0.$$

Each of $Z(B)$ and $S(Z(B))$ have $d = 0$ and all their terms in $\mathcal{B}$. So each is the direct sum (and the direct product) of complexes of the form $\cdots \to 0 \to 0 \to B_n \to 0 \to 0 \to \cdots$ with $B_n \in \mathcal{B}$ and with B_n in the n^{th} place. So we want to prove $\mathrm{Ext}^1(A, B) = 0$ when $A \in \tilde{A}$ and $B = \cdots \to 0 \to 0 \to B_n \to 0 \to \cdots$ Since $\mathrm{Ext}^1(A_k, B_n) = 0$, any exact $0 \to B \to U \to A \to 0$ splits at the module level and so can be thought of as the sequence associated with the mapping cone of a morphism $S^{-1}(A) \to B$. So by Proposition 3.3.2 we only need to prove that any such morphism is homotopic to 0. Any morphism $S^{-1}(A) \to B$ is homotopic to 0 if and only if every morphism $A \to S(B)$ is homotopic to 0. Since $S(B)$ has the same form as B, we only need argue that any $A \to B$ is homotopic to 0. But a morphism $A \to B$ is given by a commutative diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & A_{n+1} & \longrightarrow & A_n & \longrightarrow & A_{n-1} & \longrightarrow & A_{n-2} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & B_n & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

This induces a diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & Z_{n-1}(A) & \longrightarrow & A \\
 & & \downarrow & \swarrow & \\
 & & B_n & &
\end{array}$$

Since $0 \to Z_{n-1}(A) \hookrightarrow A_{n-1} \to Z_{n-2}(A) \to 0$ is exact and since

$$\mathrm{Ext}^1(Z_{n-2}(A), B) = 0,$$

we see that we get the extension $A_{n-1} \to B_n$ of $Z_{n-1}(A) \to B_n$. But this extension provides the desired homotopy $A \to B$. So we now have $\tilde{\mathcal{A}} \subset {}^\perp\tilde{\mathcal{B}} \cap \mathcal{E}$.

Now let $A \in {}^\perp\tilde{\mathcal{B}} \cap \mathcal{E}$. We want to show that $A \in \tilde{\mathcal{A}}$. Since $A \in \mathcal{E}$ (and so is exact), this means we must show that $Z_n(A) \in \mathcal{A}$. This means that we need to show that $\mathrm{Ext}^1(Z_n(A), B) = 0$ for all $B \in \mathcal{B}$. For $B \in \mathcal{B}$, let $0 \to B \to E \to B' \to 0$ be exact with E injective. Then since $(\mathcal{A}, \mathcal{B})$ is hereditary, we have that $B' \in \mathcal{B}$. So thinking of $0 \to B \to E \to B' \to 0$ as a complex with E in any position, we have a complex in $\tilde{B}$. So to get $\mathrm{Ext}^1(Z_n(A), B) = 0$, we need that $\mathrm{Hom}(Z_n(A), E) \to$

$\operatorname{Hom}(Z_n(A), B') \to 0$ is exact. Thus suppose we have a linear $Z_n(A) \to B'$. Using this we create morphism of complexes

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & A_{n+4} & \longrightarrow & A_{n+3} & \longrightarrow & A_{n+2} & \longrightarrow & A_{n+1} & \longrightarrow & A_n & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow 0 & & \downarrow 0 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & B' & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

Then since $A \in {}^{\perp}\mathcal{B}$, we use the mapping cone sequence and Proposition 3.3.2 to get that this morphism is homotopic to 0. The homotopy gives a map $A_n \to B'$ which extends the original $Z_n(A) \to B'$. But

$$\operatorname{Hom}(A_n, E) \to \operatorname{Hom}(A_n, B') \to \operatorname{Ext}^1(A_n, B) = 0$$

is exact, so the extension $A_n \to B'$ has a lifting $A_n \to E$. Hence the original $Z_n(A) \to B'$ has a lifting $Z_n(A) \to E$. Consequently $\operatorname{Ext}^1(Z_n(A), B) = 0$. So $Z_n(A) \in \mathcal{A}$ and we have that $A \in \tilde{\mathcal{A}}$. This completes the argument that $\tilde{\mathcal{A}} = {}^{\perp}\mathcal{B} \cap \tilde{\mathcal{B}}$. The argument that $\mathcal{E} \cap \tilde{\mathcal{A}} = \tilde{\mathcal{B}}$ is similar. □

7.4 More Hovey Pairs

In Section 7.3, we saw how to start with a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in R-Mod which is cogenerated by a set and then create the cotorsion pairs $(\tilde{\mathcal{A}}, \tilde{\mathcal{A}}^{\perp})$, $({}^{\perp}\mathcal{B}, \tilde{\mathcal{B}})$ in $C(R\text{-Mod})$. Then we saw that if $(\mathcal{A}, \mathcal{B})$ is hereditary, this pair of pairs in a Hovey pair. This suggests we return to the results of Section 7.2 and ask if the two pairs of cotorsion pairs we formed in $C(R\text{-Mod})$ give us a Hovey pair. The pairs we formed were $(C(\mathcal{A}), C(\mathcal{A})^{\perp}), (C(\mathcal{A}) \cap \mathcal{E}, (C(\mathcal{A}) \cap \mathcal{E})^{\perp})$, $({}^{\perp}C(\mathcal{B}), C(\mathcal{B}))$ and $({}^{\perp}(C(\mathcal{B} \cap \mathcal{E}), C(\mathcal{B}) \cap \mathcal{E})$. We want to argue that we get two associated Hovey pairs. These are $(C(\mathcal{A}) \cap \mathcal{E}, (C(\mathcal{A}) \cap \mathcal{E})^{\perp})$, $(C(\mathcal{A}), C(\mathcal{A})^{\perp})$ and $({}^{\perp}C(\mathcal{B}), C(\mathcal{B}))$, $({}^{\perp}(C(\mathcal{B}) \cap \mathcal{E}), C(\mathcal{B}) \cap \mathcal{E})$.

We will consider a single cotorsion pair in $C(R\text{-Mod})$, say $(\mathcal{C}, \mathcal{D})$, and ask when it could be the first pair or the second pair of a Hovey pair. In order to be the first pair of a Hovey pair, we would need $\mathcal{C} \subset \mathcal{E}$ and in order to be the second pair we would need $\mathcal{D} \subset \mathcal{E}$. The next lemma is concerned with the question of when $(\mathcal{C}, \mathcal{D})$ could be the second pair of Hovey pair.

Lemma 7.4.1. *Let $(\mathcal{C}, \mathcal{D})$ be a cotorsion pair in $C(R\text{-Mod})$ such that $\mathcal{D} \subset \mathcal{E}$. Then $\mathcal{D} = \mathcal{E} \cap (\mathcal{C} \cap \mathcal{E})^{\perp}$.*

Proof. We have $(\mathcal{D} \subset \mathcal{E}$ by hypothesis. Also $\mathcal{D} = \mathcal{C}^{\perp} \subset (\mathcal{C} \cap \mathcal{E})^{\perp}$. So $\mathcal{D} \subset \mathcal{E} \cap (\mathcal{C} \cap \mathcal{E})^{\perp}$. Now let $D \in \mathcal{E} \cap (\mathcal{C} \cap \mathcal{E})^{\perp}$.

We want to argue that $D \in \mathcal{D} = \mathcal{C}^{\perp}$. So if $C \in \mathcal{C}$ we need to show that $\mathrm{Ext}^1(C, D) = 0$. We now recall that $({}^{\perp}\mathcal{E}, \mathcal{E})$ is a cotorsion pair which is cogenerated by a set (see section 4.1). So this cotorsion pair is complete. This means that we have an exact sequence $0 \to C \to E \to P \to 0$ with $E \in \mathcal{E}$ and $P \in {}^{\perp}\mathcal{E}$. Since $D \subset \mathcal{E}$, we have ${}^{\perp}\mathcal{E} \subset {}^{\perp}\mathcal{D} = \mathcal{C}$. So $P \in \mathcal{C}$ and hence $E \in \mathcal{C}$ since $\mathcal{C}$ is closed under extensions. So $E \in \mathcal{C} \cap \mathcal{E}$. Since $D \in (\mathcal{C} \cap \mathcal{E})^{\perp}$, we have that $\mathrm{Ext}^1(E, D) = 0$. But $0 \to C \to E \to D \to 0$ gives the exact sequence

$$0 = \mathrm{Ext}^1(E, D) \to \mathrm{Ext}^1(C, D) \to \mathrm{Ext}^2(P, D)$$

But now $P \in {}^{\perp}\mathcal{E}$ and $D \in \mathcal{E}$. If $0 \to D \to I \to D' \to 0$ is exact with I injective, then I is exact. So D' is exact, i.e. $D' \in \mathcal{E}$. But then $0 = \mathrm{Ext}^1(P, D') = \mathrm{Ext}^2(P, D)$. So since $\mathrm{Ext}^2(P, D) = 0$, we get that $\mathrm{Ext}^1(C, D) = 0$. Thus we have established that $\mathcal{D} = \mathcal{E} \cap (\mathcal{C} \cap \mathcal{E})^{\perp}$. □

With the notation of the lemma above, we consider the prospective Hovey pair $(\mathcal{C} \cap \mathcal{E}, (\mathcal{C} \cap \mathcal{E})^{\perp}), (\mathcal{C}, \mathcal{D}) = (\mathcal{C}, \mathcal{E} \cap (\mathcal{C} \cap \mathcal{E})^{\perp})$. In fact if the first pair is a cotorsion pair, then we do have a Hovey pair.

Theorem 7.4.2. *If* $(\mathcal{A}, \mathcal{B})$ *is a cotorsion pair in* R-Mod *which is cogenerated by a set, then* $(C(\mathcal{A}) \cap \mathcal{E}, (C(\mathcal{A}) \cap \mathcal{E})^{\perp}), (C(\mathcal{A}), C(\mathcal{A})^{\perp})$ *is a Hovey pair.*

Proof. With the remark immediately preceding this theorem and using Theorem 7.2.15, we see that we only need that $C(\mathcal{A})^{\perp} \subset \mathcal{E}$. Since $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair, we have $R \in \mathcal{A}$. Hence $S^k(\underline{R}) \in C(\mathcal{A})$ for all k. Then the claim $C(\mathcal{A})^{\perp} \subset \mathcal{E}$ follows from Corollary 2.1.7. □

Dual arguments give the next result.

Theorem 7.4.3. *If* $(\mathcal{A}, \mathcal{B})$ *is a cotorsion pair in* R-Mod *which is cogenerated by a set, then* $({}^{\perp}C(\mathcal{B}), C(\mathcal{B})), ({}^{\perp}(C(\mathcal{B}) \cap \mathcal{E}), C(\mathcal{B}) \cap \mathcal{E})$ *is a Hovey pair in* $C(R$-Mod).

These two results provide us with two Hovey pairs in $C(R$-Mod) associated with the cotorsion pair $(\mathcal{A}, \mathcal{B})$ in R-Mod. In general these pairs are distinct.

7.5 Exercises

1. Prove Theorem 7.4.3.
2. Give an example where the two Hovey pairs of Theorem 7.4.2 and Theorem 7.4.3 are distinct.
3. Find an example of a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in R-Mod which is cogenerated by a set where $\tilde{\mathcal{A}} \neq C(\mathcal{A}) \cap \mathcal{E}$ (see Sections 7.2 and 7.3).

4. Let Inj $\subset R$-Mod consist of all the injective modules. Show that ${}^{\perp}C(\mathrm{Inj}) \subset \mathcal{E}$ where $\mathcal{E}$ is the class of exact complexes.

5. With the same notation as in problem 4., find an example where ${}^{\perp}C(\mathrm{Inj}) \neq \mathcal{E}$.

6. This exercise concerns the material in Section 7.2. With the notation of that section, let $\sigma = \omega$ (the least infinite ordinal number) and suppose $M = \sum_{n<\omega} A_n$ for submodules A_n of M.

   Find examples where:

   a) every $S \subset \{n|n < \omega\}$ is closed

   b) the only closed $S \subset \{n|n < \omega\}$ are $\emptyset$, $\{0\}, \{0, 1\}, \dots, \{0, 1, 2, \dots, n\}, \dots$ and $\{0, 1, 2, 3, \dots, n, \dots\}$.

Chapter 8
Minimal Complexes

8.1 Minimal Resolutions

Let $0 \to M \to E^0 \to E^1 \to \cdots$ be a minimal injective resolution of the left R-Module M, then the complex $E = \ldots 0 \to 0 \to E^0 \to E^1 \to E^2 \to \cdots$ has the property that if $f : E \to E$ is a homology isomorphism, then f induces a commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & M & \longrightarrow & E^0 & \longrightarrow & E^1 & \longrightarrow & E^2 & \longrightarrow & \cdots \\
 & & \downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & E^0 & \longrightarrow & E^1 & \longrightarrow & E^2 & \longrightarrow & \cdots
\end{array}$$

where $M \to M$ is an isomorphism. Then since we began with a minimal injective resolution of M we get that $f^0, f^1, \ldots$ are all isomorphisms. So $f : E \to E$ is an isomorphism. The same argument can be given in case we start with a minimal projective resolution (when such exists) or any kind of minimal resolution we get from a class $\mathcal{F}$ of left R-modules which is covering or enveloping (see Section 5.1 of Volume I). These examples motivate the next definition.

Definition 8.1.1. A complex $C \in C(R\text{-Mod})$ is said to be *homologically minimal* if any homology isomorphism $f : C \to C$ is an isomorphism in $C(R\text{-Mod})$.

When a ring R is local and left Noetherian, then any finitely generated left R-module M has a projective cover $P \to M$ (see Theorem 5.3.3 of Volume I). In fact P is then free and finitely generated. If $P = R^n$, then $\operatorname{Ker}(P = R^n \to M) \subset \boldsymbol{m}^n$ where $\boldsymbol{m}$ is the maximal ideal of R. Since R is left Noetherian, $\operatorname{Ker}(P \to M)$ is finitely generated and so we see that M has a minimal projective resolution

$$\cdots \to P_1 \to P_0 \to M \to 0$$

with each P_n finitely generated and free. We also have $\operatorname{Im}(P_{n+1} \to P_n) \subset \boldsymbol{m} P_n$ for $n \geq 0$.

It would be of interest to characterize the homologically trivial $P \in C(R\text{-Mod})$ where each P_n is a finitely generated free module. If we use a weaker notion of minimality then we can get a complete characterization of the minimal such P.

Definition 8.1.2. A complex $C \in C(R\text{-Mod})$ is said to be *homotopically minimal* if every homotopy isomorphism $f : C \to C$ (see Section 3.2) is an isomorphism.

Note that if $f : C \to C$ is a homotopy isomorphism, then f is a homology isomorphism. So if C is homologically minimal it is also homotopically minimal. But the converse is not true (see the exercises).

Proposition 8.1.3. *If R is a local ring with maximal ideal $\boldsymbol{m}$ and $P \in C(R\text{-Mod})$ is such that each P_n is free and finitely generated, then P is homotopically minimal if and only if* $\operatorname{Im}(P_{n+1} \to P_n) \subset \boldsymbol{m}P_n$ *for all n.*

Proof. Suppose that P satisfies the conditions and that $f : P \to P$ is a homotopy isomorphism. We want to prove that f is an isomorphism. The morphism $f : P \to P$ induces a morphism $P/\boldsymbol{m}P \to P/\boldsymbol{m}P$ which is easily seen to be a homotopy isomorphism. So $P/\boldsymbol{m}P \to P/\boldsymbol{m}P$ is a homology isomorphism. But by our condition on P, we have that $d_{n+1}(P_{n+1}) \subset \boldsymbol{m}P_n$ for each n and so the d on $P/\boldsymbol{m}P$ is 0. Hence $P/\boldsymbol{m}P \to P/\boldsymbol{m}P$ is an isomorphism. This means $P_n/\boldsymbol{m}P_n \to P_n/\boldsymbol{m}P_n$ is an isomorphism for every n. But then by Nakayama's lemma, $f_n : P_n \to P_n$ is an isomorphism and so $f : P \to P$ is an isomorphism.

Conversely, suppose P is homotopically minimal. If $d_{n+1}(P_{n+1}) \not\subseteq \boldsymbol{m}P_n$ for some n, let $y \in P_{n+1}, n+1(y) \notin \boldsymbol{m}P_n$. So $P_{n+1}, P_n \neq 0$ and y and $d_{n+1}(y)$ are parts of bases of P_{n+1} and P_n, respectively. We have the subcomplex $T = \cdots \to 0 \to Ry \to Rx \to 0 \to \cdots$ of P with $Ry \cong Rx \cong R$. So the complex T is homotopically trivial.

We now claim T is a direct summand of P. For consider the exact sequence $0 \to T \hookrightarrow P \to P/T \to 0$. By our choice of T, this sequence splits at the module level. So it can be thought of as the short exact sequence associated with a mapping cone of a morphism $g : S(P/T) \to T$. But T is homotopically trivial and so $g \cong 0$. But this gives that the sequence is split exact. So we have $P = T \oplus P/T$. But then the morphism $0 \oplus \mathrm{id}_{P/T} : P \to P$ is a homotopy isomorphism which is not an isomorphism. This contradiction shows we must have $d_{n+1}(P_{n+1}) \subset \boldsymbol{m}P_n$ for every n. □

We now would like to characterize the homotopically trivial $I \in C(R\text{-Mod})$ where each I_n is injective. The characterization and the argument are similar (in a dual sense) to that given in Proposition 8.1.3 above. However, the step corresponding to the fact that $T \subset P$ is a direct summand of P will require the next result.

Lemma 8.1.4. *If $g : C \to D$ is a morphism in $C(R\text{-Mod})$ such that the morphism $D \to C(g)$ of the exact sequence $0 \to D \to C(g) \to S(C) \to 0$ is a homotopy isomorphism, then $S(C)$ is homotopically trivial.*

Proof. We will use the construction of Section 3.4. Recall that if $X \in C(R\text{-Mod})$, then we can form the complex $\mathcal{H}om(D, X)$. This construction is functorial, so we can form the sequence

$$0 \to \mathcal{H}om(S(C), X) \to \mathcal{H}om(C(g), X) \to \mathcal{H}om(D, X) \to 0$$

of complexes. From the definition of these complexes and from the fact that $0 \to D \to C(g) \to S(C) \to 0$ splits at the module level, we see that this sequence is exact. By hypothesis $D \to C(g)$ is a homotopy isomorphism. But the $\mathcal{H}om(C(g), X) \to \mathcal{H}om(D, X)$ is also a homotopy isomorphism. But this means that $\mathcal{H}om(C(g), X) \to \mathcal{H}om(D, X)$ is a homology isomorphism. Consequently, the complex $\mathcal{H}om(S(C), X)$ is exact. Letting $X = S(C)$, we have that $\mathcal{H}om(S(C), S(C))$ is exact. Then using Proposition 3.4.3 we get that $\mathrm{id}_{S(C)} \cong 0$. By this means that $S(C)$ is homotopically trivial (see Section 3.2). □

Proposition 8.1.5. *Let $I \in C(R\text{-Mod})$ be such that I_m is an injective module for every m. Then the following are equivalent:*

1) *I is homotopically trivial*
2) *I has no nonzero factors which are homotopically trivial*
3) *$Z_n(I) \subset' I_n$ (i.e. I_n is an essential extension of $Z_n(I)$) for each n.*

Proof. 1) ⇒ 2) If $I = I' \oplus I''$ where $I' \neq 0$ and where I' is homotopically trivial, then as in the proof of Proposition 8.1.3 we can produce a homotopy isomorphism $f : I \to I$ that is not an isomorphism.

2) ⇒ 3). If I_n is not an essential extension of $Z_n(I)$, let $Z_n(I) \cap U = 0$ for a submodule $U \subset I_n$, $U \neq 0$. Then $d_n \mid U$ is an injection. Since $U \subset I_n$ and since I_n is injective we can find $V \subset I_n$ with $U \subset' V$ and V injective. Then since $d_n|U$ is an injection we get $d_n|V$ is also an injection. So we have the subcomplex $\cdots \to 0 \to V \xrightarrow{\cong} d_n(V) \to 0 \to \cdots$ of I. But from Section 1.4 we see that this complex is injective. So it is a direct summand of I. But it is homotopically trivial. This contradicts 2).

3) ⇒ 1). Let f be a homotopy isomorphism. We first prove that $\mathrm{Ker}(f) = 0$. Let g give a homotopy inverse of f. So $g \circ f \cong \mathrm{id}_C$. It suffices to prove that $\mathrm{Ker}(g \circ f) = 0$. For $n \in \mathbb{Z}$ let $K_n = \mathrm{Ker}((g \circ f)_n)$ and let $L_n = K_n \cap Z_n(I)$. Then $L_n \subset' K_n$. Now suppose $g \circ f \overset{s}{\cong} \mathrm{id}_C$. So $\mathrm{id}_C - g \circ f = d \circ s + s \circ d$.

If we apply this equation to $x \in L_n$ and use the fact that $x \in \mathrm{Ker}((g \circ f)_n) \cap \mathrm{Ker}(d_n)$, we get that $x = d_{n+1}(s_n(x))$. So $s_n|L_n$ and $d_{n+1}|s_n(L_n)$ are injections. Hence $s_n(L_n) \cap Z_{n+1}(I) = 0$. But then, by 3), $s_n(L_n) = 0$. So $L_n = 0$ and thus since $L_n \subset' K_n$, we have that $K_n = 0$.

So now we have $\mathrm{Ker}(g \circ f) = 0$ and so that $\mathrm{Ker}(f) = 0$. Hence we have a short exact sequence $0 \to I \xrightarrow{f} I \to \mathrm{Coker}(f) \to 0$. Since I_n is injective for each n, we have that this sequence splits at the module level and so can be thought of as a short exact sequence associated with a mapping cone. By Lemma 8.1.4, we get that $\mathrm{Coker}(f)$ is homotopically trivial. Now we appeal to Proposition 3.3.3. Note that since f is a homotopy isomorphism, $[f]$ clearly admits a retraction (and in fact an inverse) in $K(R\text{-Mod})$. This means we can write $I = I' \oplus \mathrm{Coker}(f)$ (where $I' \cong I$). Since $\mathrm{Coker}(f)$ is homotopically trivial, we can create a homotopy isomorphism $g : I \to I$ that is 0 on $\mathrm{Coker}(f)$. But from the above, we know $\mathrm{Ker}(g) = 0$. Hence $\mathrm{Coker}(f) = 0$ and so f is an isomorphism. □

Lemma 8.1.6. *Let $C, D \in C(R\text{-Mod})$ be homotopically minimal. If $f : C \to D$ is a homotopy isomorphism, then f is an isomorphism.*

Proof. If $g : D \to C$ is such that $[g] = [f]^{-1}$, then g is also a homotopy isomorphism. So $g \circ f : C \to C$ is a homotopy isomorphism. Thus $g \circ f$ is an isomorphism since C is minimal. Similarly $f \circ g$ is an isomorphism. Hence f (and g) are isomorphisms. □

8.2 Decomposing a Complex

Proposition 8.2.1. *If $I \in C(R\text{-Mod})$ is such that I_n is injective for every n, then $I = I' \oplus I''$ where I' is homotopically minimal and where I'' is homotopically trivial. Furthermore if $I = J' \oplus J''$ is another such decomposition, then $I' \to I \to J'$ is an isomorphism where both maps $I' \to I$ and $I \to J'$ come from the decompositions above.*

Proof. For each n we pick a maximal submodule $U_n \subset I_n$ such that $Z_n(I) \cap U_n = 0$. Then U_n is injective and we have the injective subcomplex $\cdots \to 0 \to U_n \xrightarrow{\cong} d_n(U_n) \to 0 \to \cdots$ of I. The sum of these subcomplexes is direct. So if I is this sum, then I is injective. So I is a direct summand of I. If $I = I' \oplus I''$, then by the construction of I'' we get that I'' is homotopically minimal (by (3) of Proposition 8.1.5).

So we have the desired decompositions $I = I' \oplus I'$. If $J = J' \oplus J'$ is another such decomposition, then $I' \to I$ and $I \to J'$ are homotopy isomorphism since I' and J'' are homotopically trivial. Hence the composition $I' \to I \to J'$ is a homotopy isomorphism. So $I' \to J'$ is a homotopy isomorphism. Thus by Lemma 8.1.6, $I' \to J'$ is an isomorphism. □

8.3 Exercises

1. a) Find a module M having a submodule S such that $\mathrm{Hom}(M/S, M) = 0$ and such that there is an endomorphism $f : M \to M$ with $f(S) \subset S$ such that $S \to S$ is an automorphism of S but such that $f : M \to M$ is not an automorphism of M.

   b) Let $C = \cdots \to 0 \to M \to M/S \to 0 \to 0 \to \cdots$ (with M in the 0^{th} place). Argue that C is homotopically trivial but not homologically trivial.

2. If C and C' in $C(R\text{-Mod})$ are homologically (homotopically) minimal, show that $C \oplus C'$ is homologically (homotopically) minimal. Hint. Look at exercise 3 of Chapter 4 of Volume I.

3. If R is a left perfect ring and $P \in C(R\text{-Mod})$ is such that all P_n are projective, prove that P has a direct sum decomposition $P = P' \oplus P''$ where P' is homotopically minimal and where P'' is homotopically trivial.

4. If $\cdots \to P_1 \to P_0 \to M \to 0$ is a projective resolution of the left R-module M, let $P = \cdots \to P_1 \to P_0 \to 0 \to \cdots$. If P has a direct sum decomposition $P = P' \oplus P''$ as in 3 above, argue that $\cdots \to P'_2 \to P'_1 \to M \to 0$ is a minimal projective resolution of M.

5. Let $P \in C(R\text{-Mod})$ be such that every P_n is finitely generated and projective. Let P^* be the complex $\cdots \to P^*_{-1} \to P^*_0 \to P^*_1 \to \cdots$ where $P^*_n = \mathrm{Hom}(P_n, R)$ is the algebraic dual of P_n. Argue that P is homotopically minimal if and only if P^* is homotopically minimal.

6. Prove that a ring R is left perfect if and only if every $P \in C(R\text{-Mod})$ with all P_n projective has a direct sum decomposition $P = P' \oplus P''$ with P' homotopically minimal and P'' homotopically trivial.

7. Find examples of short exact sequences $0 \to C' \to C \to C'' \to 0$ in $C(R\text{-Mod})$ where C' and C'' are homotopically trivial but where C is not homotopically trivial.

Chapter 9
Cartan and Eilenberg Resolutions

A *Cartan and Eilenberg projective resolution* of a complex C is a certain exact sequence of complexes

$$\cdots \to P^{-1} \to P^0 \to C \to 0$$

To define these resolutions we need to begin with the notion of a Cartan–Eilenberg projective complex. Then we will consider an injective version of these notions.

9.1 Cartan–Eilenberg Projective Complexes

Definition 9.1.1. A complex $P \in C(R\text{-Mod})$ is said to be *Cartan–Eilenberg projective complex* (hereafter called a *C–E projective complex*) if P, $Z(P)$, $B(P)$, and $H(P)$ all have each of their terms a projective module.

Remarks and Examples 9.1.2. A direct sum of a family of complexes is C–E projective if and only if each of the summands is C–E projective. For any $k \in \mathbb{Z}$, $S^k(\bar{R})$ and $S^k(\underline{R})$ are C–E projective. Hence any free, and so any projective complex is C–E projective. Also any complex P with all P_n projective and such that $d^P = 0$ is C–E projective.

The next result shows that from these observations we can find all C–E projective complexes.

Proposition 9.1.3. *A complex $P \in C(R\text{-Mod})$ is a C–E projective complex if and only if P has a direct sum decomposition $P = P' \oplus P''$ where P' is a projective complex and where P'' has all its terms projective and* 0 *differential (i.e. $d^{P''} = 0$).*

Proof. Assume $P = P' \oplus P''$ with P' projective and where each P''_n is projective and $d^{P''} = 0$. Then by the remarks above P is C–E projective.

Conversely, assume P is C–E projective. For every n we have the exact sequence

$$0 \to Z_n(P) \to P_n \to B_{n-1}(P) \to 0$$

and

$$0 \to B_n(P) \to Z_n(P) \to H_n(P) \to 0.$$

Since $B_{n-1}(P)$ and $H_n(P)$ are projective modules, these sequences are split exact.

So using these splittings, we get $Z_n(P) = B_n(P) \oplus H_n(P)$ and $P_n = Z_n(P) \oplus B_{n-1}(P)$. Combining these we get $P_n = B_n(P) \oplus H_n(P) \oplus B_{n-1}(P)$. Then with these decompositions we get that $d_n : P_n = B_n(P) \oplus H_n(P) \oplus B_{n-1}(P) \to P_{n-1} = B_{n-1}(P) \oplus H_{n-1}(P) \oplus B_{n-2}(P)$ is the map $(x, y, z) \mapsto (z, 0, 0)$. As a consequence, we see that $P = P' \oplus P''$ where P' is the direct sum of the complexes

$$\cdots \to 0 \to (0 \oplus 0 \oplus B_{n-1}(P)) \to (B_{n-1}(P) \oplus 0 \oplus 0) \to 0 \to \cdots$$

and where P'' is the direct sum of the complexes $\cdots \to 0 \to H_n(P) \to 0 \to \cdots$. So using the description of the projective complexes in Chapter 1, we see that P' is projective. Hence we have the desired direct sum decomposition.

We now want to argue that every $C \in C(R\text{-Mod})$ has a C–E projective precover (see Chapter 5 of Volume I). To do so, first note that if P'' has $d^{P''} = 0$, then a morphism $P'' \to C$ (for any $C \in C(R\text{-Mod})$) has a morphism $P'' \to Z(C)$. If $P'' \to Z(C)$ is an epimorphism, Q'' has $d^{Q''} = 0$ and $Q'' \to C$ is a morphism, then the diagram

$$\begin{array}{ccc} & & Q'' \\ & \overset{P}{\swarrow} & \downarrow \\ P'' & \longrightarrow & Z(C) \end{array}$$

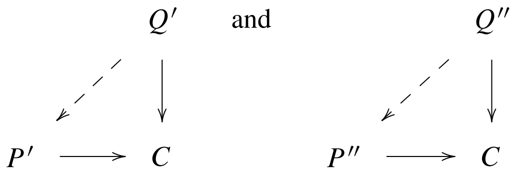
can be completed to a commutative diagram, and so

$$\begin{array}{ccc} & & Q'' \\ & \overset{P}{\swarrow} & \downarrow \\ P'' & \longrightarrow & C \end{array}$$

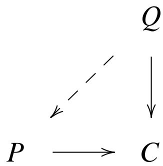
can be completed to a commutative diagram. □

Proposition 9.1.4. *Every* $C \in C(R\text{-Mod})$ *has a C–E projective precover.*

Proof. Let $\varphi' : P' \to C$ be an epimorphism where P' is a projective complex and let $\varphi'' : P'' \to C$ be such that each P''_n is projective, $d^{P''} = 0$, and such that the corresponding $P'' \to \mathbf{Z}(C)$ is an epimorphism. Let $\varphi : P' \oplus P'' \to C$ agree with φ' on P' and with φ'' on P''. Let $\psi : Q \to \mathbf{C}$ be any morphism where Q is C–E projective. Write $Q = \mathbf{Q}' \oplus Q''$ where Q' is projective and where $d^{Q''} = 0$. Then each of the diagrams



can be completed to commutative diagrams. Hence so can



□

Note that this precover is an epimorphism and so every precover is an epimorphism.

9.2 Cartan and Eilenberg Projective Resolutions

Definition 9.2.1. If $C \in C(R\text{-Mod})$, then by a *Cartan and Eilenberg projective resolution* (or a C–E projective resolution) of C we mean a complex of complexes

$$\cdots \to P^{-2} \to P^{-1} \to P^0 \to C \to 0$$

where each P^{-n} is C–E projective and where $P^0 \to C$, $P^{-1} \to \text{Ker}(P^0 \to C)$ and $P^{-n} \to \text{Ker}(P^{-n+1} \to P^{-n+2})$ for $n \geq 2$ are C–E projective precovers.

Every complex C has such a resolution, and we get the usual comparison results concerning such resolutions.

The next result will be used to give a characterization of these resolutions.

Lemma 9.2.2. *Let* $0 \to A' \to A \to A'' \to 0$ *be a complex of complexes in* $C(R\text{-Mod})$. *Consider the complexes:*

1) $0 \to A' \to A \to A'' \to 0$
2) $0 \to Z(A') \to Z(A) \to Z(A'') \to 0$
3) $0 \to B(A') \to B(A) \to B(A'') \to 0$
4) $0 \to A'/Z(A') \to A/Z(A) \to A''/Z(A'') \to 0$
5) $0 \to A'/B(A') \to A/B(A) \to A''/B(A'') \to 0$
6) $0 \to H(A') \to H(A) \to H(A'') \to 0$

If 1) *and* 2) *are exact, then all of* 1)–6) *are exact. If* 1) *and* 5) *are exact, then all of* 1)–6) *are exact.*

Proof. Assume 1) and 2) are exact. Apply the snake lemma of Section 2.3 to the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z(A') & \longrightarrow & Z(A) & \longrightarrow & Z(A'') & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0
\end{array}$$

This gives 4).

Recall that we can consider $d : A \to S(A)$ as a morphism of complexes. This induces an isomorphism $A/Z(A) \to B(S(A))$. So using this we see that 3) $\Leftrightarrow$ 4). Then we can again use the snake lemma to get 5) and 6).

If 1) and 5) are exact, then the snake lemma gives 3) is exact. Then using the snake lemma on

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
 & & \big\downarrow d' & & \big\downarrow d & & \big\downarrow d'' & & \\
0 & \longrightarrow & B(S(A')) & \longrightarrow & B(S(A)) & \longrightarrow & B(S(A'')) & \longrightarrow & 0
\end{array}$$

we get 2) exact. So then as above we get 1)–6) are all exact. □

Proposition 9.2.3. *Let Q be a C–E projective complex in $C(R\text{-Mod})$ and let $Q \to C$ be a morphism in $C(R\text{-Mod})$ with kernel K. Then the following are equivalent*

a) *$Q \to C$ is a C–E projective precover*

b) $0 \to K \to Q \to C \to 0$ *and* $0 \to Z(K) - Z(Q) \to Z(C) \to 0$ *are exact*

c) *for any* $\mathbf{k} \in \mathbb{Z}$*, any diagrams*

$$\begin{array}{ccc}
 & & S^k(\bar{R}) \\
 & \swarrow & \big\downarrow \\
Q & \longrightarrow & C
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
 & & S^k(\underline{R}) \\
 & \swarrow & \big\downarrow \\
Q & \longrightarrow & C
\end{array}$$

can be completed to commutative diagrams.

Furthermore, when these conditions hold we get

$$0 \to B(K) \to B(Q) \to B(C) \to 0$$
$$0 \to K/Z(K) \to Q/Z(Q) \to C/Z(C) \to 0$$
$$0 \to K/B(K) \to Q/B(Q) \to C/B(C) \to 0$$
$$0 \to H(K) \to H(Q) \to H(C) \to 0$$

are exact.

Proof. First note that if $Q \to C$ is a C–E projective precover, then $Q \to C \to 0$ is exact and so $0 \to K \to Q \to C \to 0$ is exact. And if this sequence is exact, so is $0 \to Z(K) \to Z(Q) \to Z(C)$.

Hence we see that the equivalence of b) and c) is given by Lemma 9.2.2.

Using the isomorphisms $\mathrm{Hom}(S^k(\underline{R}), C) \cong Z_{-k}(C)$ and $\mathrm{Hom}(S^k(\bar{R}), C) \cong C_{-k+1}$, we see that c) $\Leftrightarrow$ b). We get c) $\Leftrightarrow$ a) by using Proposition 9.1.3 and the observation that (with the notation of that result) that we have $P = P' \oplus P'' \subset F = F' \oplus F''$ with P' a direct summand of F'' where F' is free and F'' has all its terms F''_n free and where $d^{F''} = 0$. So then F' is a direct sum of copies of $S^k(\bar{R})$'s and F'' of copies of $S^k(\underline{R})$'s. The last claim follows from Lemma 9.2.2 above. □

Theorem 9.2.4. *If* $\cdots \to P^{-2} \to P^{-1} \to P^0 \to C \to 0$ *is a complex of complexes in* $C(R\text{-Mod})$ *where each* P^{-n} *is C–E projective, then it is a C–E projective resolution of* C *if and only if it is exact and* $\cdots \to Z(P^{-2}) \to Z(P^{-1}) \to Z(P^0) \to Z(C) \to 0$ *is exact. When this is the case,*

1) $\cdots \to B(P^{-2}) \to B(P^{-1}) \to B(P^0) \to B(C) \to 0$,

2) $\cdots \to P/Z(P^{-1}) \to P^0/Z(P^0) \to C/Z(C) \to 0$,

3) $\cdots \to P^{-1}/B(P^{-1}) \to P^0/B(P^0) \to C/B(C) \to 0$ *and*

4) $\cdots \to H(P^{-1}) \to H(P^0) \to H(C) \to 0$

are also exact.

Proof. These claims follow from Proposition 9.2.3 above and the definition of a C–E projective resolution. □

We note that when we have such a resolution as in the above, then at the module level, we get the projective resolutions

$$\cdots \to P^1_n \to P^0_n \to C_n \to 0$$
$$\cdots \to Z_n(P^1) \to Z_n(P^0) \to Z_n(C) \to 0$$

and so forth.

9.3 C–E Injective Complexes and Resolutions

The definitions and results of this section are dual to those of Sections 9.1 and 9.2.

Definition 9.3.1. A complex $I \in C(R\text{-Mod})$ is a *C–E injective complex* if I, $Z(I)$, $B(I)$ and $H(I)$ all have each of their terms an injective module.

A product of complexes is C–E injective if and only if each of the complexes is C–E injective. Any direct summand of a C–E injective complex is C–E injective.

Proposition 9.3.2. *A complex $I \in C(R\text{-Mod})$ is C–E injective if and only if I has a direct sum decomposition $I = I' \oplus I''$ where I' is injective and I'' has all its terms injective modules and $d^{I''} = 0$.*

Proof. Dual to that of Proposition 9.1.3. □

Proposition 9.3.3. *Every $C \in C(R\text{-Mod})$ has a C–E injective preenvelope.*

Proof. If $C \to I'$ and $C/B(C) \to I''$ are monomorphisms where I' is an injective complex and I'' has all its terms injective and $d^{I''} = 0$, then $C \to I' \oplus I''$ is a C–E injective preenvelope. □

A *C–E injective resolution* of $C \in C(R\text{-Mod})$ is defined in the obvious manner.

Theorem 9.3.4. *If $0 \to C \to I^0 \to I^1 \to \cdots$ is a complex of complexes in $C(R\text{-Mod})$ where each I^n is C–E injective, then it is a C–E injective resolution of C if and only if $0 \to C \to I^0 \to I^1 \to \cdots$ and $0 \to C/B(C) \to I^0/B(I^0) \to I^1/B(I^1) \to \cdots$ are exact. When this is the case, we get all four of the other related sequences exact, namely that the following*

1) $0 \to Z(C) \to Z(I^0) \to Z(I^1) \to \cdots$
2) $0 \to B(C) \to B(I^0) \to B(I^1) \to \cdots$
3) $0 \to C/Z(C) \to I^0/Z(I^0) \to I^1/Z(I^1) \to \cdots$
4) $0 \to H(C) \to H(I^0) \to H(I^1) \to \cdots$

are exact.

Proof. We only indicate the modifications we need to carry out the duals of the corresponding result concerning a C–E projective resolution. We first note that if I is an injective module, then $S^k(\bar{I})$ and $S^k(\bar{I})$ are C–E injective complexes. For $C \in C(R\text{-Mod})$, we have isomorphisms $\mathrm{Hom}(C, S^k(\bar{I})) \cong \mathrm{Hom}(C_{k+1}, I)$ and $\mathrm{Hom}(C, S^k(\underline{I})) \cong \mathrm{Hom}(C_k/B_k(C), I)$. With these observations the proof can be completed. □

9.4 Cartan and Eilenberg Balance

The balance in this section is that of Chapter 8 of Volume I.

Proposition 9.4.1. *Let* $0 \to K \to P \to C \to 0$ *be exact in* $C(R\text{-Mod})$ *where* $P \to C$ *is a C–E projective precover of* C*. If* $I \in C(R\text{-Mod})$ *is C–E injective, then* $0 \to \mathrm{Hom}(C, I) \to \mathrm{Hom}(P, I) \to \mathrm{Hom}(K, I) \to 0$ *is exact.*

Proof. We use the decomposition $I = I' \oplus I''$ of Proposition 9.3.2. Then using the structure theorem for injective complexes (see Chapter 1), we see that we only need to prove that if $I \in R$-Mod is an injective module then $\mathrm{Hom}(-, S^k(\bar{I}))$ and $\mathrm{Hom}(-, S^k(\underline{I}))$ each leaves the sequence $0 \to K \to P \to C \to 0$ exact. But then we use the isomorphisms given at the end of the proof of Theorem 9.3.4 and see that the claim follows from the exactness of $0 \to K \to P \to C \to 0$ and $0 \to K/B(K) \to P/B(K) \to C/B(C) \to 0$, i.e, from the exactness of $0 \to K_n \to P_n \to C_n \to 0$ and of

$$0 \to K_n/B_n(K) \to P_n/B_n(P) \to C_n/B_n(C) \to 0$$

for each $n \in \mathbb{Z}$. □

The dual result with a dual proof is:

Proposition 9.4.2. *Let* $0 \to D \to I \to C \to 0$ *be exact in* $C(R\text{-Mod})$ *where* $D \to I$ *is a C–E injective preenvelope of* D*. If* $P \in C(R\text{-Mod})$ *is C–E projective, then* $0 \to \mathrm{Hom}(P, D) \to \mathrm{Hom}(P, I) \to \mathrm{Hom}(P, C) \to 0$ *is exact.*

The two results give us the main result of this section.

Theorem 9.4.3. *The functor* $\mathrm{Hom}(-,-)$ *on* $C(R\text{-Mod}) \times C(R\text{-Mod})$ *is right balanced by* C–E Proj × C–E Inj *where* C–E Proj *is the class of C–E projective complexes and* C–E Inj *is that of the C–E injective complexes.*

9.5 Exercises

1. Let $P = P' \oplus P'' = Q' \oplus Q$" be two decompositions of a C–E projective complex P as in Proposition 9.1.3. Argue that $P'' \to P' \oplus P'' = Q' \oplus Q'' \to Q''$ is a homology isomorphism. So deduce that $P'' \to Q''$ is an isomorphism.
2. Let $D \in C(R\text{-Mod})$ have $d^D = 0$. Show that D has a C–E injective envelope $D \to I$ and that $d^I = 0$.
3. Let $0 \to C' \to C \to C'' \to 0$ be an exact sequence in $C(R\text{-Mod})$. Prove that $0 \to \mathrm{Hom}(P, C') \to \mathrm{Hom}(P, C) \to \mathrm{Hom}(P, C'') \to 0$ is exact for every C–E projective complex P if and only if $0 \to Z(C') \to Z(C) \to Z(C'') \to 0$ is exact.

4. If $0 \to C' \to C \to C'' \to 0$ in $C(R\text{-Mod})$ is exact and if $0 \to Z(C') \to Z(C) \to Z(C'') \to 0$ is exact, prove the C–E projective version of the horse shoe lemma.
5. Let $P \in C(R\text{-Mod})$ and suppose that $B(P)$ and $H(P)$ both have all their terms projective. Argue that P is a C–E projective complex.
6. Given $P = P' \oplus P'' \to C$ in $C(R\text{-Mod})$ where P' is projective and P'' has all its terms projective and $d^{P''} = 0$. If $P'' \to Z(C)$ and $P' \to C \to C/Z(C)$ are both epimorphisms, argue that $P \to C$ is a C–E projective precover.
7. Let $(C^i)_{i \in I}$ be a family of complexes in $C(R\text{-Mod})$. Show that there are C–E projective precovers $P^i \to C^i$ such that $\bigoplus_{i \in I} P^i \to \bigoplus_{i \in I} C^i$ is a C–E projective precover. Deduce that if $Q^i \to C^i$ is a C–E projective precover for each $i \in I$ then $\bigoplus_{i \in I} Q^i \to \bigoplus_{i \in I} C^i$ is also a C–E projective precover.
8. If $P, E \in C(R\text{-Mod})$ where P is C–E projective and E is exact prove that $\text{Ext}^1(P, E) = 0$. Then argue that if E is such that $\text{Ext}^1(P, E) = 0$ for all C–E projective complexes P, then E is exact.
9. Prove that a product of C–E injective preenvelopes is a C–E injective preenvelope.

Bibliographical Notes

Chapter 1. The material in this chapter is standard. The characterizations of the projective and injective complexes are well known but perhaps hard to find.

Chapter 2. The interpretation of elements of $\mathrm{Ext}^1(M, N)$ (where M and N are modules) as short exact sequences can be found in MacLane [19]. The arguments are categorical and so carry over to more general abelian categories. Much more information on mapping cones can be found in Verdier's thesis [22].

Chapter 3. Most of the material in this chapter is standard, but the approach owes much to Bourbaki [2]. This is background material for reading about triangulated and derived categories. The splitting results appeared in [9] and [3]. The Koszul section (i.e. Section 3.5) can be found in Greenlees and Dwyer [14].

Chapter 4. Cotorsion pairs were introduced by Salce [21] in the category of abelian groups. The terminology seems to have come from Harrison's thesis [15]. Interest in them was stirred by the Eklof–Trlifaj paper [6] and by applications in categories of complexes (Enochs, Jenda, Xu [9]). The Dold triplet (but not with that terminology) appeared in that paper, but it was Greenlees and Dwyer's work [14] that suggested the results in Section 4.2.

Chapter 5. The topic of this chapter is from Neeman [20] but the approach is that of Bravo, Enochs, Jacob, Jenda, and Rada.

Chapter 6. The material in this chapter can be found in Hovey [17].

Chapter 7. The treatment of the Hill lemma is from Göbel and Trlifaj [13]. Much of the rest of the chapter is based on work of Gillespie (see [11] and [12]).

Chapter 8. The fact that minimal projective and injective resolutions give rise to homologically minimal complexes was noticed once the existence of these minimal resolutions was observed. The first definition of a homotopically minimal complex seems to have been given in Enochs, Jenda, Xu [9]. In that paper there was an early version of Proposition 8.1.5 (Proposition 3.15 of [9]). The complete version is due to Krause [18]. Proposition 8.14 is from Avramov and Martsinkovsky [1].

Chapter 9. These resolutions were introduced in the last chapter of Cartan and Eilenberg ([4]). Verdier gave the definition of a Cartan–Eilenberg injective complex in his thesis ([22], Definition 4.6.1).

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