TOPOLOGICAL SIX FUNCTORS, AND REROLLEMENT OF CATEGORIES

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Contents

1. Algebraic Structures and Categorical Setup

1.1. The Forgetfulness.

Slogan. The idea of algebraic structures arises from forgetfulness.

Example 1.1 ($\twoheadrightarrow \circ \hookrightarrow$)**.** Every set maps $f : X \to Y$ admits a "canonical" decomposition

$$X \twoheadrightarrow \operatorname{coim}(f) \simeq \operatorname{im}(f) \hookrightarrow Y. \tag{1.1}$$

In language of CAT, X and Y are discrete categories. By observation:

(1) f is surjective as a map $\iff f$ is essentially surjective as a functor,
(2) f is injective as a map $\iff f$ is full as a functor.

Slogan. Injection is actually surjection "in higher dimension".

Example 1.2 (Composition of Functors)**.** There are three kinds of "surjections" for functor $F : \mathcal{C} \to \mathcal{D}$.

$$\text{(su)} : \text{ essentially surjective}, \quad \text{(fu)} : \text{ full}, \quad \text{(fa)} : \text{ faithful}. \tag{1.2}$$

Hence, any functor decomposes into three parts:

$$\mathcal{C} \xrightarrow{\text{su, fu, not fa}} \operatorname{im}^2(F) \xrightarrow{\text{su, not fu, fa}} \operatorname{im}^1(F) \xrightarrow{\text{not su, fu, fa}} \mathcal{D}. \tag{1.3}$$

For set maps, $\mathcal{C} \to \operatorname{im}^2(F)$ is an isomorphism.

Proposition 1.3. *The decomposition is unique under the equivalences of categories.*

Definition 1.4 (Forgetfulness)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be functor.

(1) Say F forget only attachments, whence F is su and fu;
(2) Say F forget only structures, whence F is su and fa;
(3) Say F forget only properties, whence F is fu and fa.

Remark 1.5. There are no inherent issues in defining previously undefined terms; however, it is not always practical or meaningful to introduce such definitions. In what follows, we will illustrate through a few examples that our proposed definition is reasonable.

Example 1.6 (The forgetfulness of only attachments)**.** Let Vect be the categories of "vector spaces with ambient field", wherein the objects are the pair $(\mathbb{F}, V_{\mathbb{F}})$, and the morphism are commutative squares

$$\begin{array}{ccc} \mathbb{K} & \overset{\text{extension}}{\rightarrowtail} & \mathbb{F} \\ \downarrow & & \downarrow \\ \mathrm{End}_{\mathbb{K}}(U_{\mathbb{K}}) & \longrightarrow & \mathrm{End}_{\mathbb{F}}(V_{\mathbb{F}}) \end{array} . \tag{1.4}$$

The functor "projecting the first entry" $\mathsf{Vect} \to \mathsf{Field}$ is su, fu, yet not fa.

Example 1.7 (The forgetfulness of only structures)**.** A typical kind of examples: functors sending concrete categories into underlying sets. For instance,

$$\mathsf{Ab}_{\text{finite}} \to \mathsf{Sets}_{\text{finite and non-empty}} \tag{1.5}$$

sends finite Abelian groups to their underlying sets, which is su, fa, yet not fu.

Slogan. If $F : \mathcal{C} \to \mathsf{Sets}$ forgets no attachments, then different morphisms are different set maps.
For such F, one can simply view $\mathsf{Mor}(\mathcal{C})$ as set maps. However, F is not always compatible with universal properties, i.e. limits and colimits indexed by sets.

Example 1.8 (The forgetfulness of only properties)**.** Under equivalences, such functors are essentially the inclusion of full subcategories. For instance, the inclusion $\mathsf{Ab} \to \mathsf{Grp}$ forgets the commutativity of group product, which is fu, fa, yet not su.

Example 1.9 (decomposition of free functor)**.** Let $F : \mathsf{Set} \to \mathsf{Mod}_R$, serving as the left adjoint to the usual forgetful functor $U : \mathsf{Mod}_R \to \mathsf{Set}$. Now the decomposition of F is

$$\mathsf{Set} \to \mathsf{Set} \simeq (\mathsf{Mod}_R)' \to (\mathsf{Mod}_R)'' \to \mathsf{Mod}_R. \tag{1.6}$$

Here $(\mathsf{Mod}_R)'$ is equivalent to Set. The objects are of the form $R^{\oplus S}$ and morphisms are induced by the change of indices, that is,

$$[f : S \to T] \Rightarrow [R^{\oplus S} \to R^{\oplus T}, \quad (r_s)_{s\in S} \mapsto (r_{f(s)})_T]. \tag{1.7}$$

Now $(\mathsf{Mod}_R)''$ is the full subcategory of Mod_R generated by objects in $(\mathsf{Mod}_R)'$.

Remark 1.10. It is not so amazing that free functors are compositions of surjections (quotients), since $x \in \emptyset$ satisfies any propositions.

1.2. On Limits and Colimits.

Slogan. Universal property is exactly the initial objects or terminal objects in certain category. In short, universal properties are nothing but limits or colimits, indexed by certain diagrams (small categories).

Example 1.11. For instance, the coproduct of a set of objects e.g. $X \coprod Y$, is the is exactly the initial object in the category $\mathcal{D}$, wherein
(1) $\mathsf{Ob}(\mathcal{D}) = \{(E, f, g) \mid f : X \to E \leftarrow Y : G\}$;
(2) the morphisms are exactly the commutative diagrams of the form

$$\begin{array}{ccccc} X & \xrightarrow{f'} & E' & \xleftarrow{g'} & Y \\ \| & & \downarrow \ell: & & \| \\ X & \xrightarrow[f]{} & E & \xleftarrow[g]{} & Y \end{array} . \tag{1.8}$$

Remark 1.12. Our convention of universal properties is compatible with most of the cases, since most of the property is determined by a set indexed generators with relations; whereas the "universal property" of localisation comes from categorical equivalent relations (usually a filtered colimit index by proper classed).

Definition 1.13 (limits)**.** Let $I : \mathcal{B} \to \mathcal{C}$ be a diagram, i.e. functor from small category $\mathcal{B}$.
(1) Consider the co-cones with fixed base $I(\mathcal{B})$ and a collection of flexible generatrices $\{\alpha_b : I(b) \to c\}_{b\in\mathcal{B}}$. Whence there is an initial object, i.e. the generatrices $\{\alpha_b^0 : I(b) \to c^0\}_{b\in\mathcal{B}}$ which any $\{\alpha_b : I(b) \to c\}_{b\in\mathcal{B}}$ factors through in a unique way, we say $c^0 := \varinjlim I$ (along with the generatrices) a colimit of I.
(2) The limit is exactly the opposite statement for cones.

Once we view cones (co-cones) as degenerate cylinders, generatrices are exactly the natural transformations between I and the constant functor, i.e. $\{\alpha_b : I(b) \to c\}_{b\in\mathcal{B}} \in (I, \underline{c})_{\mathrm{Funct}(\mathcal{B},\mathcal{C})}$. The "$\exists$ unique" statements are better replaced by representable functors, i.e.

$$(\varinjlim I, c)_{\mathcal{C}} \simeq (I, \underline{c})_{\mathrm{Funct}(\mathcal{B},\mathcal{C})}, \quad \varphi \mapsto \{\varphi \circ [I(b) \to \varinjlim I]\}_{b\in\mathcal{B}} \tag{1.9}$$

Hence, the adjoint triple writes $(\varinjlim \dashv \underline{(\cdot)} \dashv \varprojlim)$.

Example 1.14. Consider the basic symbols in a category: start of an arrow s, terminal of an arrow t, along with $\mathrm{id}_\bullet$ establish the adjoint triple $t \dashv \mathrm{id}_\bullet \dashv s$, where $\mathrm{id}_\bullet$ sends $\mathcal{C}$ to the morphism category $\mathcal{C}^{\to}$. Such adjoint triple also admits an explanation in language of limits, and so is its generation to simplicial categories.

Definition 1.15 (The preserving, reflecting, and creating)**.** Let $I : \mathcal{B} \to \mathcal{C}$ be a small diagram, $F : \mathcal{C} \to \mathcal{D}$ be a functor. Set $\lim \in \{\varinjlim, \varprojlim\}$.

(1) (preserve) Say F preserves lim, whenever $(c = \lim I) \implies (F(c) = \lim(F \circ I))$;
   It also means that F commutes with the lim.

(2) (reflect) Say F reflect lim, whenever $c \neq \lim I \Rightarrow (F(c) \neq \lim(F \circ I))$
   It also means that only lim maps to lim.

(3) (create) Say F creates lim, whenever $\lim(F \circ I)$ is always of the form $F(\lim I)$.

Example 1.16. Set lim = ker, $F : \mathcal{C} \to \mathcal{D}$ are additive functors between Abelian categories.

(1) Say F preserves all kernels, whence $0 \to K \to X \to Y$ in $\mathcal{C}$ yields $0 \to F(K) \to F(X) \to F(Y)$ in $\mathcal{D}$;

(2) Say F reflects all kernels, whence $0 \to F(K) \to F(X) \to F(Y)$ in $\mathcal{D}$ yields $0 \to K \to X \to Y$ in $\mathcal{C}$;

(3) Say F creates all kernels, whence $0 \to M \to F(X) \to F(Y)$ in $\mathcal{D}$ is exactly the image of some $0 \to K \to X \to Y$ (where $M = F(K)$).

Proposition 1.17. *Here are some feasible propositions on limits.*

(1) *By commutativity of left adjoints (resp. right adjoints), colimits (resp. limits) commutes.*

(2) *By currying (and Yoneda lemma), the canonical isomorphism holds for locally small categories*

$$(\varinjlim -, \cdot) \simeq \varprojlim(-, \cdot), \quad (\cdot, \varprojlim) \simeq \varprojlim(\cdot, -). \tag{1.10}$$

(3) *For inclusion of small diagrams $i : I_0 \hookrightarrow I$ with the same set of vertices, the pre-composition $i^* : \mathrm{Funct}(I, \mathcal{C}) \to \mathrm{Funct}(I_0, \mathcal{C})$ creates all lim. An application: the lim of chain complexes is constructed component-wise, wherein differentials are induced by universal property of limits.*

(4) *Fully faithful functors (functors forgets only propositions) reflects all lim.*

Example 1.18 (Why limits and filtered colimits?)**.** Limits and filtered colimits meets logical issues. For most of the concrete categories, e.g. Sets, Group, Ab, Ring,

(1) the limit object is a subset of the product object, wherein $x \in \varprojlim X_i$ usually understood as the collection of elements in X_i satisfying several requirements, or just a function in $I \to X$;

(2) the colimit object is a quotient subset of co-product object, wherein $x \in \varinjlim X_i$ understood as the equivalence classes whose base set is a disjoint union $\bigsqcup X_i$;

(3) the filtered colimit is a special type of colimit, where one can distinguish whether finitely many objects are in the same equivalent class without taking colimit object.

Slogan. Limits and filtered colimits are "logical" in our common sense.

Proposition 1.19 (exchanging lims)**.** *Let $F : I \times J \to \mathcal{C}$ be a bi-functor from diagrams I and J.*

(1) *There is an canonical morphism $\varinjlim_{i \in I} \varprojlim_{j \in J} F(i, j) \to \varprojlim_{j \in J} \varinjlim_{i \in I} F(i, j)$.*

The canonical morphism is more used for posets, known as the "min-max" inequality.

(2) *If J is finite, I is filtered, and $\mathcal{C} = \mathsf{Sets}$, then the canonical morphism is an isomorphism.*

Remark 1.20. Assume that $\mathcal{C}$ has filtered colimits, finite limits, and there is some functor $U : \mathcal{C} \to \mathsf{Sets}$ preserving filtered colimits, finite limits, and reflect isomorphisms. In particular, assume $U : \mathcal{C} \to \mathsf{Sets}$ provides an algebraic structure. In this case, filtered colimits commutes with finite limits.

Slogan. Filtered colimits is exact for algebraic structures.

Remark 1.21. A categorical restatement of filtered colimits $\varinjlim_{i\in I} X_i$ is that any finite subdiagram admits a cocone (not necessary to be the colimit).
To be explicit, let $X : I \to \mathcal{C}$ be a functor. For any finite subdiagram $I_0 \subseteq I$, the system $X(I_0)$ factors through some $X(i_0)$.

Proposition 1.22. *Let I be a filtered system, and $X : I \to \mathcal{C}$ is the functor to concrete category*[1]*. Suppose that $\{\iota_i : Z \hookrightarrow X_i\}_{i\in I}$ is a set of monomorphisms, then $f : Z \hookrightarrow \varinjlim X$ is also a monomorphism. The simple observation: in order verify $(f(z_1) = f(z_2)) \Rightarrow (z_1 = z_2)$, one can firstly find $x_i \in X_i$ which maps to $f(z_i)$ in the system. Since the system generated by $\{X_1, X_2\}$ factors through some X_0, where $f(z_1) = f(z_2)$ maps from the same element in X_0. Since $Z \hookrightarrow X_0$ is injective, $z_1 = z_2$ is clear.*

Example 1.23. By experiences, there are some forgetful functor creates limits and filtered colimits (e.g. groups to sets, compact Hausdorff spaces to topological spaces, Modules to their underlying Abelian groups, Sheaves to presheaves over Noetherian sites).

Remark 1.24. We highlight that the limit/colimit of the empty diagram is the terminal/initial object.

[1] For formal and general statement, one can assume that $\mathcal{C}$ has generator(s) G, where "$x \in X$" is formalised by a morphism $x \in (G, X)_{\mathcal{C}}$

1.3. What are and Why Algebraic Structures?

Definition 1.25 (Forgetful functors from algebraic structures to sets)**.** An algebraic structure is a category over Sets via a functor $U : \mathcal{C} \to \mathsf{Sets}$, provides

(1) U is faithful, i.e. if $f, g \in \mathsf{Mor}(\mathcal{C})$ are different morphisms, then $U(f)$ and $U(g)$ are different maps between underlying sets;
 In other words, U does not forget the attachments "underlying set".
(2) U reflect isomorphisms, i.e. $U(f)$ is bijective between sets whenever f is an isomorphism;
 A trivial fact: functors preserves isomorphisms.
(3) U commutes with limits and filtered colimits (if exists).
 In other words: one can calculate sub-object of products, or equivalences classes in set level.
(4) For most of the cases, we requires $\mathcal{C}$ to have limits and filtered colimits (indexed by sets).

Here $U : \mathcal{C} \to \mathsf{Sets}$ (or simply $\mathcal{C}$) is called an algebraic structure.

Example 1.26. Examples and non-examples.

- $\mathsf{Sets}_\bullet$ (sets with a base point) and Grp are algebraic structures;
- Mon (monoid with unit) and Ring are algebraic structures;
- Alg_k, $\mathsf{CommAlg}_k$, Lie_k are algebraic structures (k is commutative ring);
- Mod_R, and in particular Ab, are algebraic structures.

Recall the definition of algebraic structures. The following non-examples are worth mentioning.

- Top does not satisfies (2): let (X, τ_1) and (X, τ_2) be topological spaces with $\tau_1 \subseteq \tau_2$. The trivial map $(X, \tau_2) \to (X, \tau_1)$ is bijective on sets, yet not always the homeomorphism.
- $\mathsf{Ring}^{\mathrm{op}}$ does not satisfies (3), since U does not preserves colimits (e.g., $\mathrm{coker}(\mathbb{Z} \hookrightarrow \mathbb{Q})$) in general.

Proposition 1.27 (Major Properties of Alegbraic Structures)**.** *Let $U : \mathcal{C} \to \mathsf{Sets}$ be an algebraic structure.*

(1) *$U(-)$ reflects monomorphism, epimorphism, isomorphisms, and retractions.*

The key observation is that $U(-)$ reflects the (left or right) cancelling properties.

(2) *$U(-)$ preserves isomorphisms, monomorphisms, yet not all epimorphisms.*

Functors preserves isomorphisms. $\mathbb{Z} \to \mathbb{Q}$ is epimorphism yet not surjection.

For monomorphisms, notice that $f : X \to Y$ is a monomorphism whenever the induced map $X \times_Y X \to X$ is an isomorphism, and $X \times_Y X$ is a kind of limit.

Proposition 1.28 (sub-codomain)**.** *We see that $U(-)$ preserves and reflects pull-back squares. Since*

$$\begin{array}{ccc} & & X \\ & \swarrow & \downarrow f \\ Y & \xrightarrow[i]{} & Z \end{array} \quad \Longleftrightarrow \quad \begin{array}{ccc} X \times_Z Y & \xrightarrow{\text{split epi.}} & X \\ \downarrow & PB & \downarrow f \\ Y & \xrightarrow[i]{} & Z \end{array}, \tag{1.11}$$

that is, $X \times_Z Y \to X$ is an split epimorphism whenever f factor through i. We see that

(1) *If $\mathcal{C}$ has pull-backs, then $U(-)$ reflects all factorisation conditions.*

(2) *When i is an monomorphism, $X \times_Z Y \simeq Y$. Hence, $f : X \to Z$ factor through the sub-object Y, whenever $\operatorname{im}(f) \subseteq Y$. It illustrates that one can take the sub-codomain if necessary.*

2. Sheaves on Algebraic Structures

2.1. Presheaves over Algebraic Structures.

Definition 2.1 (The category of open sets)**.** Let (X, τ) be the usual topological space. Define the category of open sets $\mathsf{Open}(X)$ with $\mathsf{Ob}(\mathsf{Open}(X)) = \tau$, and

$$(U, V)_{\mathsf{Open}(X)} = \begin{cases} i_{V,U} & U \subseteq V, \\ \emptyset & U \not\subseteq V. \end{cases} \tag{2.1}$$

$\mathsf{Open}(X)$ is also a modular lattice closed under finite $\wedge$ and arbitrary $\vee$.

Remark 2.2. Continuous map $f : X \to Y$ gives $f^{-1} : \mathsf{Open}(Y) \to \mathsf{Open}(X)$.
By observation, $f^{-1} : \mathsf{Subset}(Y) \to \mathsf{Subset}(X)$ admits
(1) a left adjoint $f : \mathsf{Subset}(X) \to \mathsf{Subset}(Y)$, i.e. $f(A) \subseteq B \Longleftrightarrow A \subseteq f^{-1}(B)$;
(2) a right adjoint $c \circ f \circ c : \mathsf{Subset}(X) \to \mathsf{Subset}(Y)$, i.e. $f(A^c) \subseteq B^c \Longleftrightarrow A^c \subseteq f^{-1}(B^c)$.
By since left (resp. right) adjunction commutes with colimit (resp. limit), one has

$$f(\bigcup_{i\in I} U_i) = \bigcup_{i\in I} f(U_i), \quad f(\bigcap_{i\in I} U_i) \subseteq \bigcap_{i\in I} f(U_i), \tag{2.2}$$

$$f^{-1}(\bigcup_{i\in I} U_i) = \bigcup_{i\in I} f^{-1}(U_i), \quad f^{-1}(\bigcap_{i\in I} U_i) = \bigcap_{i\in I} f^{-1}(U_i). \tag{2.3}$$

Some issues should be further discussed in the study of open and closed embedding.

Definition 2.3 (Presheaves)**.** Given $\mathsf{Open}(X)$, the category of pre-sheaves is exactly the functor category

$$\mathrm{PSh}(X) := \mathrm{Funct}(\mathsf{Open}(X)^{\mathrm{op}}, \mathsf{Sets}). \tag{2.4}$$

An element $s \in F(V) =: \Gamma(V, F)$ is called a section on V. The morphisms are

$$F(i_{V,W}) := \mathrm{Res}_{W,V} : F(V) \to F(W). \tag{2.5}$$

Remark 2.4. In our convention, the morphisms are indexed by $f_{j,i} : A_i \to A_j$.

Definition 2.5 (Presheaves in Algebraic Structures)**.** Let $U : \mathcal{C} \to \mathsf{Sets}$ be an algebraic structure. Set $\mathrm{PSh}_{\mathcal{C}}(X) := \mathrm{Funct}(\mathsf{Open}(X)^{\mathrm{op}}, \mathcal{C})$. Clearly, the composition gives $U \circ (-) : \mathrm{PSh}_{\mathcal{C}}(X) \to \mathrm{PSh}(X)$.

Definition 2.6 (Direct Image)**.** For continuous map $f : X \to Y$, one has $f^{-1} : \mathsf{Open}(Y) \to \mathsf{Open}(X)$. Direct image is defined as the pre-composition of functor

$$f_* : \mathrm{PSh}_{\mathcal{C}}(X) \to \mathrm{PSh}_{\mathcal{C}}(Y), \quad F \mapsto F \circ f^{-1}. \tag{2.6}$$

Remark 2.7. The composition is covariant: $(f \circ g)_* F = F \circ (f \circ g)^{-1} = F \circ g^{-1} \circ f^{-1} = f_* g_* F$.

Definition 2.8 (Inverse image of presheaves)**.** $f_p : \mathrm{PSh}_{\mathcal{C}}(Y) \to \mathrm{PSh}_{\mathcal{C}}(X)$ is defined as the left adjoint of direct image f_*. For arbitrary $F \in \mathrm{PSh}_{\mathcal{C}}(X)$ and $G \in \mathrm{PSh}_{\mathcal{C}}(Y)$, one has

$$\begin{aligned}
(G, f_* F)_{\mathrm{PSh}_{\mathcal{C}}(Y)} &\simeq \{G(V) \to F(f^{-1}(V))\}_{V \in \mathsf{Open}(Y)} && (2.7)\\
&\simeq \{G(V) \to F(U) \mid U \subset f^{-1}(V)\}_{U \in \mathsf{Open}(X), V \in \mathsf{Open}(Y)} && (2.8)\\
(\text{by } f \dashv f^{-1}) \quad &\simeq \{G(V) \to F(U) \mid f(U) \subset V\}_{U \in \mathsf{Open}(X), V \in \mathsf{Open}(Y)} && (2.9)\\
&\simeq \left(\varinjlim_{f(\bullet) \subset V} G(V), F(\bullet) \right)_{\mathrm{PSh}_{\mathcal{C}}(X)} =: (f_p G, F)_{\mathrm{PSh}_{\mathcal{C}}(X)}. && (2.10)
\end{aligned}$$

The final colimit is filtered, thus exists in $\mathcal{C}$, and commutes with U.

Remark 2.9. By adjunction, the composition is contravariant $(g \circ f)_p = f_p \circ g_p$.

Slogan. We can calculate both f_* and f_p on sets.

Example 2.10 (Analysis of $f_p \dashv f_*$)**.** The unit of the adjunction is $\mathrm{id}_{\mathrm{PSh}_{\mathcal{C}}(Y)} \to f_* f_p$. For any $G \in \mathrm{PSh}_{\mathcal{C}}(Y)$ and $V \in \mathsf{Open}(Y)$, one has

$$(f_* f_p G)(V) = (f_p G)(f^{-1}(V)) = \varinjlim_{f(f^{-1}(V)) \subset W} G(W). \tag{2.11}$$

If we further assume that f is an surjection, one has $f(f^{-1}(V)) = V$, and thus η is an isomorphism. The counit is $f_p f_* \to \mathrm{id}_{\mathrm{PSh}_{\mathcal{C}}(X)}$. For any $F \in \mathrm{PSh}_{\mathcal{C}}(X)$ and $U \in \mathsf{Open}(X)$, one has

$$(f_p f_* F)(U) = \varinjlim_{f(U) \subset V} (f_* F)(V) = \varinjlim_{U \subset f^{-1}(V)} F(f^{-1}(V)) \xrightarrow{\text{finer system}} \varinjlim_{U \subset W} F(W) = F(U). \tag{2.12}$$

Whenever $f^{-1} : \mathsf{Open}(Y) \to \mathsf{Open}(X)$ is a surjection. The counit is an isomorphism.

Example 2.11 (Stalks, and Skyscraper sheaves)**.** Let $f : \{*\} \simeq \{x\} \subseteq X$ be continuos map. The stalk functor is exactly $f_p : \mathrm{PSh}_{\mathcal{C}}(X) \to \mathrm{PSh}_{\mathcal{C}}(\{*\}) \simeq \mathcal{C}$, i.e., for any $F \in \mathrm{PSh}_{\mathcal{C}}(X)$,

$$(f_p F)(\{*\}) = \varinjlim_{f(*) \subseteq W} F(W) = \varinjlim_{x \in W} F(W) = F_x. \tag{2.13}$$

The pull-back presheaf f_* is called skyscraper presheaf. Let $A \in \mathcal{C}$ be an object in algebraic structure, then

$$(f_* A)(U) = A(f^{-1}(U)) = \begin{cases} A & x \in U, \\ \top \text{ (termianl object)} & x \notin U. \end{cases} \tag{2.14}$$

Notation. We usually assume that $F(\emptyset) = \top$ (terminal) for presheaves. It is harmless to deal with presheaves without this property (one has to make this "assumption" for sheaves!).

Remark 2.12. We identify the above map with its image, i.e. $x : \{*\} \to X, \quad * \mapsto x$.

(1) The unit sends F to the skyscraper sheaf at x with value F_x.

(2) The counit is tautological, since x^{-1} is an surjection.
(3) If we introduce another point $y \in X$, then the "merged" counit is

$$y_p x_* A = \varinjlim_{y \in W} A_{\text{whether } x \in W} = A_{\text{whether } y \in \overline{\{x\}}} = A_{\text{whether } \overline{\{y\}} \subseteq \overline{\{x\}}}. \tag{2.15}$$

$y_p x_* : \mathcal{C} \to \mathcal{C}$ is identical whenever $\overline{\{y\}} \subseteq \overline{\{x\}}$; and send any object to $\emptyset$ otherwise.
We say y is a specialisation of x (resp. x is a generalisation of y) whence $\overline{\{y\}} \subseteq \overline{\{x\}}$

Example 2.13 (Global Sections, Constant Presheaf)**.** Let $p : X \twoheadrightarrow \{*\}$ be the canonical surjection. The global section $\Gamma(X, F) := F(X)$ coincides $p_* F$, i.e., $(p_* F)(\{*\}) = F(X)$.
The constant presheaf $\underline{A}$ is defined as inverse image $p_p A \in \mathrm{PSh}_{\mathcal{C}}(X)$, i.e,

$$(p_p A)(U) = \varinjlim_{U \subseteq p^{-1}(*)} A(*) = \begin{cases} A & U \neq \emptyset, \\ \top & Y = \emptyset. \end{cases} \tag{2.16}$$

Remark 2.14. The unit of p is identical, the counit is to take constant sheaf with respect to the global section.

2.2. Sheafification (General Approach).

Example 2.15 (2 bad examples of presheaves)**.** Let B^A be functions of type $A \to B$. By practice,

(1) when a function is defined on every single point in A, it belongs to B^A;
(2) a function is uniquely determined by its values on every single point.

Let X be a topological space which is not very simple, e.g. $\mathbb{R}$ with standard topology.

(1) Let F be the presheaf of bounded continuous functions, i.e. sending each open set U to $C_b^0(U)$. For $U \subseteq V$, $\mathrm{Res}_{U,V}$ is the usual restriction map $C_b^0(V) \to C_b^0(U), \quad f \mapsto f|_U$.
Why it is bad? Any unbounded continuous function $f \in C^0(\mathbb{R})$ is glued by a collection sections, which agree on their intersections; however, $f \notin F(\mathbb{R})$. Some sections should be added.
(2) Let F be the presheaf taking value 0 on any proper open subset of $\mathbb{R}$, while $F(\mathbb{R}) \neq 0$.
Why it is bad? For any distinct $s, t \in F(\mathbb{R})$, the sections agree on any proper subset of $\mathbb{R}$. There is no need to distinguish them. Some sections should be removed.

Definition 2.16 (Two criterions of being sheaves)**.** In contrast to the "bad examples", we say $F \in \mathrm{PSh}(X)$ is a sheaf, provided the following two criteria.

(1) (unity criterion) Let U be an open set with open cover $\{U_i\}_{i \in I}$. We assume $U = \bigcup_{i \in I} U_i$ for simplicity. The unity criterion says that
$$\iota : F(U) \to \prod_{i \in I} F(U_i), \quad s \mapsto (\mathrm{Res}_{U_i,U}(s))_{i \in I} \tag{2.17}$$
is always a monomorphism for arbitrary choice of $U = \bigcup_{i \in I} U_i$. In short, a global section is uniquely determined by restrictions via arbitrary chosen open cover of it domain.
(2) (glueing criterion) Let U be an open set with open cover $U = \bigcup_{i \in I} U_i$. The glueing says that, if for any pair $(i, j) \in I \times I$, there is a $(s_i \in F(U_i))_{i \in I}$ such that $\mathrm{Res}_{U_i \cap U_j, U_i}(s_i) = \mathrm{Res}_{U_i \cap U_j, U_j}(s_j)$, then there

is a unique $s \in F(U)$ such that $\iota(s) = (s_i)_{i\in I}$. In language of category, the following is an equaliser of parallel lines:

$$F(U) \xrightarrow{(\mathrm{Res}_{U_k,U})_{k\in I}} \prod_{k\in I} F(U_k) \underset{\prod_{k\in I}(\prod_{i\in I}\mathrm{Res}_{U_i\cap U_k,U_k})}{\overset{\prod_{k\in I}(\prod_{j\in I}\mathrm{Res}_{U_k\cap U_j,U_k})}{\rightrightarrows}} \prod_{(i,j)\in I\times I} F(U_i\cap U_j)\ . \tag{2.18}$$

In short, a collection of compatible sections glue to a global section.

Definition 2.17 (Category of sheaf)**.** Say $F \in \mathrm{PSh}_{\mathcal{C}}(X)$ is a sheaf, whenever the above two criteria holds. Write $F \in \mathrm{Sh}_{\mathcal{C}}(X)$. Since sheaves are presheaves with certain properties, the forgetful functor, the morphisms between sheaves are just the morphisms between underlying presheaves, i.e. $\mathrm{Sh}_{\mathcal{C}}(X) \hookrightarrow \mathrm{PSh}_{\mathcal{C}}(X)$ is fully faithful.

Remark 2.18. Since $U : \mathcal{C} \to \mathsf{Sets}$ commutes with products and reflect the pull-back diagrams, $F \in \mathrm{PSh}_{\mathcal{C}}(X)$ is a sheaf whenever $U \circ F \in \mathrm{PSh}_{\mathsf{Sets}}(X)$.

Remark 2.19. In additive cases, the two criteria jointly establish the left exact sequence

$$0 \to \underbrace{F(U)}_{\text{unity criterion}} \xrightarrow{(\mathrm{Res}_{U_k,U})_{k\in I}} \underbrace{\prod_{k\in I} F(U_k)}_{\text{gluing criterion}} \xrightarrow{\text{Up - Down}} \prod_{(i,j)\in I\times I} F(U_i\cap U_j). \tag{2.19}$$

For each $\underbrace{A}_{P}$, one has $P \iff$ exactness at A.

Remark 2.20. (Very important) $F(\emptyset) = F(\varprojlim_{\emptyset} \cdot) = \varprojlim_{\emptyset} F(\cdot) = \top$ is the terminal object in the category.

Definition 2.21 (Topological Basis)**.** Say $\mathcal{B} \subseteq \tau$ is a basis of topology space (X, τ), whenever

(1) any $x \in X$ admits a cover $x \in U \in \mathcal{B}$;

(2) for any $U_1, U_2 \in \mathcal{C}$ with non-empty intersection $U_1 \cap U_2 \neq \emptyset$, one has $U_3 \in \mathcal{B}$ such that $U_3 \subseteq U_1 \cap U_2$.

Remark 2.22. Any $U \in \tau$ admits a $\mathcal{B}$-cover $\{U_i \in \mathcal{B}\}_{i\in I}$ such that $U = \bigcup_{i\in I} U_i$. For categorical convention, $\bigcup_{\emptyset} = \emptyset$ and $\bigcap_{\emptyset} = X$. Topological basis provides a simpler cofinal system: For any x, we have the cofinal systems $\{U \in \mathcal{B} \mid x \in U\}$ and $\{U \in \tau \mid x \in U\}$.

Proposition 2.23 (Sheaf condition for topological basis)**.** *Let $\mathcal{B}$ be the basis of (X, τ). The presheaf $F \in \mathrm{PSh}_{\mathcal{C}}(X)$ is a sheaf, whenever the following two criteria holds*

(1) *(unity criterion for topological basis) For any $U \in \tau$, and any open covering $U = \bigcup_{i\in I} B_i$, such that $B_i \cap B_j = \bigcap_{k\in I_{i,j}} B_{i,j,k}$, whenever there is a collection of sections $\{s_i \in F(B_i)\}_{i\in I}$ satisfying the following 3-cycle equality*
$$(s_i)|_{B_{i,j,k}} = (s_j)|_{B_{i,j,k}} \quad (\forall k \in I_{i,j}), \tag{2.20}$$
then there exists a unique global section $s \in F(U)$ such that $s|_{B_i} = s_i$

Slogan. $\mathrm{Sh}_{\mathcal{C}}(X, \tau) \simeq \mathrm{Sh}_{\mathcal{C}}(X, \mathcal{B})$ is an equivalence of categories.

Definition 2.24 (Seperable conditions)**.** Let $F \in \mathrm{PSh}_{\mathcal{C}}(X)$ be a presheaf.

(1) Say F is separable on level of open sets, whenever
$$F(U) \hookrightarrow \prod_{i\in I} F(U_i), \quad s \mapsto (\mathrm{Res}_{U_i,U}(s))_{i\in I} \tag{2.21}$$
is an injection for any open cover $U = \bigcup_{i\in I} F(U)$.

(2) Say F is separable on stalk level, whenever
$$F(U) \hookrightarrow \prod_{p\in U} F_p, \quad s \mapsto ([s]_p, \text{ image of the stalk at } p)_{p\in U} \tag{2.22}$$
is an injection for any open subset U.

Remark 2.25. The sheafification over topological spaces is usually done on the stalk level, whereas the sheafification over sites ("the point-free topology" defined by axioms of open sets) is done on open set levels.

Definition 2.26 (0-th Cech cohomololy)**.** Unwinding the definition of Cech cohomology in general, we define

$$\check{H}^0(\mathcal{U}, F) := \left\{ (s_i)_{i\in I} \in \prod_{i\in I} F(U_i) \mid s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j} \right\}, \tag{2.23}$$

which reads as the zeroth Cech cohomology of presheaf F with respect to the open covering $\mathcal{U}(U; \{U_i\}_{i\in I})$.

Proposition 2.27 (Restatement of sheaf condition)**.** *Consider the canonical injection*

$$F(U) \to \check{H}^0(\mathcal{U}(U; \{U_i\}_{i\in I}), F), \quad s \mapsto \Big(\mathrm{Res}_{U_i,U}(s)\Big)_{i\in I}. \tag{2.24}$$

By definition, the morphism is an isomorphism for any $\mathcal{U}$ whenever F is a sheaf.

Definition 2.28 (The $(-)^+$ functor)**.** The zeroth cech cohomology measures "how many compatible sections there are". Since open sets admits finite intersection, the poset of open covering of U with "refinement" as its partial order, denoted by $\mathcal{J}_U$, admits a filtered structure. Set

$$(-)^+ : \mathrm{PSh}_{\mathcal{C}}(X) \to \mathrm{PSh}_{\mathcal{C}}(X), \quad F \mapsto [F^+ : U \mapsto \varinjlim_{\mathcal{J}_U^{\mathrm{op}}} \check{H}^0(\mathcal{U}, F)]. \tag{2.25}$$

The definition of functor $(-)^+$ is determined by only the structure of $\mathsf{Open}(X)$, thus is functorial.

Remark 2.29. There is an amazing lemma one may deduce in the verification. Recall that the bi-functor $\check{H}^0(-,-)$ defined on Covering$(X)^{\mathrm{op}} \times \mathrm{PSh}_{\mathcal{C}}(X)$. Set the forgetful functor R : Covering$(X) \to X$, i.e. the projection $\mathcal{U}(U; \{U_i\}_{i\in I}) \to U$ (forget attachments!). Observation: $\check{H}^0$ defines on $(\mathrm{im}^2(R))^{\mathrm{op}} \times \mathrm{PSh}_{\mathcal{C}}(X)$. That is, for any morphism of the covering $\Phi(\varphi; -) : \mathcal{U}(U; -) \to \mathcal{V}(U; -)$, $\check{H}(\Phi, -)$ depends on $\check{H}(\varphi, -)$.

Proposition 2.30. $(-)^+$ *is defined by filtered colimits, hence commutes with colimits and finite limits.*

Remark 2.31. In particular, $(-)^+$ is an exact functor.

Example 2.32. Functorially, $(-)^+$ makes every presheaf separable on level of open sets.
To see that $F^+(-)$ is separable on on level of open sets, we take any two section $s, t \in F^+(U)$ factoring through some $\mathcal{U}_s, \mathcal{U}_s \in \mathcal{J}_U$, respectively. Suppose that (s) and (t) agrees in $\prod_{k\in K} F^+(U_k)$. The three coverings $\{U_k\}$, $\mathcal{U}_s$ and $\mathcal{U}_s$ admits a common refinement (here the filtered structure occurs), s coincides t in $\breve{H}^0(\mathcal{U}_{\text{common refinement}}, F)$. Hence, $s = t$.

Proposition 2.33. *If G is a presheaf separable on level of open sets, then $G \to G^+$ is induced by monomorphisms $\{G \hookrightarrow \breve{H}(\mathcal{U}, G)\}_{\mathcal{U}\in\text{Coverings}}$. Since $\mathcal{U}$'s forms a filtered system and monomorphisms coincides injections of sets, the induced map $G \hookrightarrow G^+$ is a monomorphism.*
Moreover, G^+ is a sheaf. It suffices to show the monomorphism $G^+(U) \to \breve{H}^0(\mathcal{U}, G^+)$ for any $\mathcal{U}$.
(1) *(The criterion of glueing) The surjective part follows from the definition.*
(2) *(The criterion of unity) The injective part follows from that filter colimit is exact on sets, while the underlying functor U commutes with filtered colimits and reflect monomorphisms.*

Example 2.34. We shall show in a trivial example that how $F \to F^+ \to F^{++} =: F^{\#}$[2] works. Let (X, τ) be a topological space with $X = \{a, b, c\}$ and non-empty open sets $\boxed{a\ \boxed{\boxed{c}\ \boxed{e}}}$. Set

$$F(\emptyset) = 0, \quad F(\boxed{c}) = F(\boxed{e}) = \mathbb{Q}, \quad F(\text{other open sets}) = \mathbb{Z}. \tag{2.26}$$

Separating the sheaves on level of open sets, one has

$$F^+(\emptyset) = 0, \quad F^+(\boxed{c}) = F^+(\boxed{e}) = \mathbb{Q}, \quad = F^+(\boxed{c\ e}) = \mathbb{Q}\oplus\mathbb{Q}, \quad F(\text{other open sets}) = \mathbb{Z}. \tag{2.27}$$

The final step of sheafification gives

$$F^{\#}(\emptyset) = 0, \quad F^{\#}(\boxed{c}) = F^{\#}(\boxed{e}) = \mathbb{Q}, \quad F^{\#}(\text{other open sets}) = \mathbb{Q}\oplus\mathbb{Q}. \tag{2.28}$$

[2] The notation come from $+\&+ \Rightarrow \#$

2.3. Properties of Sheafification.

Proposition 2.35. *Sheafification is functorial construction by applying* $(-)^+$ *twice. When* F *is already a sheaf, it follows from* $F(U) = \check{H}^0(\mathcal{U}, F)$ *that* $F = F^+$*, and thus* $F^\# = F$*.*
Hence, let G *be a presheaf and* F *is any sheaf, one has* $(G^\#, F) \simeq (G, F)$ *via pre-composing* $G \to G^\#$*.*

Slogan. The sheafification is a left adjoint of inclusion $\mathrm{Sh}_{\mathcal{C}}(X) \to \mathrm{PSh}_{\mathcal{C}}(X)$.

Proposition 2.36 (Sheaves and limits)**.** *The right adjoint of* $(-)^\#$ *creates all small limits.*

(1) *(Preserving) Right adjoint functor preserves all limits.*
(2) *(Reflecting) Fully faithful inclusion reflects all limits (and colimits).*
(3) *(Creating) Let* $F : I \to \mathrm{Sh}_{\mathcal{C}}(X)$ *be a diagram of sheaves. We shall show that the limit of* F *in* $\mathrm{PSh}_{\mathcal{C}}(X)$ *is exactly the limits in the sheaves category. Let* $G \in \mathrm{Sh}_{\mathcal{C}}(X)$ *be any sheaf, one has*

$$(G, \varprojlim F)_{\mathrm{PSh}} \simeq \varprojlim (G, F)_{\mathrm{PSh}} \simeq \varprojlim (G, F)_{\mathrm{Sh}}. \tag{2.29}$$

It suffices to show that $\varprojlim F$ *is indeed a sheaf. Recall that our two criteria correspond to equalisers and monomorphisms which* $\varprojlim$ *commute with. Hence,* $\varprojlim F$ *satisfies sheaf conditions.*

Slogan. Limits of sheaves are just limits of their underlying presheaves, as what "creating" exactly means.

Remark 2.37. The right adjoint of $(-)^\#$ reflects all colimits (fully-faithfulness), but does not preserves colimits in general. By adjunction, we can only deduce that

$$\varinjlim_{i \in I}{}^{\mathrm{Sh}} F_i \simeq \left(\varinjlim_{i \in I}{}^{\mathrm{PSh}} F_i \right)^\# \tag{2.30}$$

For non-examples that $\varinjlim_{i\in I}{}^{\mathrm{PSh}} F_i$ is not a sheaf, consider the following exact sequence of sheaves

$$0 \to \underbrace{2\pi i\mathbb{Z}_{\mathbb{C}^*}}_{\text{constant Sh on } \mathbb{C}^*} \overset{\iota}{\hookrightarrow} \underbrace{\mathcal{O}^{\mathrm{Hol}}_{\mathbb{C}^*}}_{\text{holomorphic functions}} \xrightarrow{\exp(-)} \underbrace{\mathcal{O}^{\mathrm{Hol,Inv}}_{\mathbb{C}^*}}_{\text{invertible holomorphic functions}} \to 0. \tag{2.31}$$

Here $\exp(-)$ is an epimorphism but not a surjection of presheaves. Hence, $\mathrm{coker}^{\mathrm{PSh}}(\iota)$ is not a sheaf.

Proposition 2.38. *It the structure $\mathcal{C}$ is Abelian, then so is $\mathrm{Sh}_{\mathcal{C}}(X)$. The additivity of functors, categorical kernel and cokernels are clear. The key part is to show that* $\mathrm{im} \simeq \mathrm{coim}$.
For morphism of sheaves $\varphi : F \to G$, one has

$$\mathrm{im}^{\mathrm{Sh}}(\varphi) = \ker(G \to (\mathrm{coker}(\varphi))^{\#}), \quad \mathrm{coim}^{\mathrm{Sh}}(\varphi) = (\mathrm{coker}(\ker\varphi))^{\#}. \tag{2.32}$$

Apply sheafification functor to the short exact sequence

$$0 \to \underbrace{\ker(\mathrm{coker}^{\mathrm{PSh}}\varphi)}_{=\mathrm{coker}^{\mathrm{PSh}}(\ker\varphi)} \to G \to \mathrm{coker}(\varphi) \to 0 \tag{2.33}$$

One see that $\mathrm{coim}^{\mathrm{Sh}}(\varphi)$ is the kernel of $G \to \mathrm{coker}(\varphi)^{\#}$. Hence, $\mathrm{im} = \mathrm{coim}$ *for Abelian categories.*

Remark 2.39. Abelian algebraic structures $\mathcal{C}$ have limits (AB3) and filtered colimits are exact (AB5).

3. Topological Six Functors

3.1. Direct image $(-)_*$ and inverse image $(-)^{-1}$.

Example 3.1 (Easy sheafification)**.** In retrospect to our general approach of sheafification, we firstly check separable conditions on level of open sets, then take the filtered colimits of $\widetilde{H}^0$ as "the finest open cover".

- What if we check separable conditions on level of stalks, which is already a filtered colimits?

It is somehow convenient to separate a section into "glueing atoms" with respect to each $p \in X$. Set

$$\Pi : \mathrm{PSh}_{\mathcal{C}}(X) \to \mathrm{PSh}_{\mathcal{C}}(X), \quad F \mapsto \left[\Pi F : V \mapsto \prod_{p \in V} F_p, \quad s \mapsto ([s]_p)_{p \in V} \right]. \tag{3.1}$$

The functor Π sends every presheaf to what takes germs (glueing atoms) of the sections. For instance,

(1) our first bad example is $C_b^0(-)$, where no information is lost in $(\Pi C_b^0)(-)$;
(2) our second bad example sends every $V \subsetneq \mathbb{R}$ to $\top$ while $F(\mathbb{R}) \neq \top$, $(\Pi F) = \top$ vanishes everywhere.

According to the previous definition, F is separable on level of stalks whence $F \hookrightarrow \Pi F$ is an injection. Now we claim that the "easy sheafification" $F(-)^{\#}$ is constructed by taking functorial subsets in ΠF; alternatively, we can deal with the construction from what we have already deduced, that is

- ΠF is a sheaf, and the induced morphism $F(-)^{\#} \to \Pi F$ is an injection.

One can either verify it with our two criteria, or simply deduce from the filtered system of injections.

Remark 3.2. Π is defined with filtered colimits, thus is exact.

Proposition 3.3. *Let $\varphi : F \to G$ be a morphism between sheaves. The following are equivalent.*

(1) *φ is a monomorphism (isomorphism) of presheaves,*
(2) *$\varphi_V : F(V) \to V(V)$ is a monomorphism (isomorphism) for any open set V,*
(3) *φ is a monomorphism (isomorphism) of sheaves,*

(4) $\varphi_x : F_x \to G_x$ *is a monomorphism (isomorphism) for any point* p.
For epimorphisms, (1) $\iff$ (2)$\Rightarrow$(3) $\iff$ (4).
(1) $\iff$ (2) *is due to the elementary properties of functor categories.*
(3) $\iff$ (4) *follows from the exactness of filtered colimits.*
(2) $\Rightarrow$ (3) *follows from the fact that* $U : \mathcal{C} \to \mathsf{Sets}$ *reflects cancelling rules and isomorphisms.*
For monomorphism, (3) $\Rightarrow$ (2) *is due to "*$i : \mathrm{Sh} \hookrightarrow \mathrm{PSh}$ *creates all small limits".*
For isomorphisms, (3) $\Rightarrow$ (2) *is due to fully-faithfulness of* i.

Remark 3.4. Nevertheless, we can restate (4) for epimorphisms on level of $(-)^+$, that is,

- for any $s \in G(V)$, there exists $V = \bigcup_{i \in I} V_i$ such that every $\mathrm{Res}_{V_i,V}(s) \in G(V_i)$ is in the image of φ_{V_i}.

Definition 3.5 (Direct image for sheaves)**.** Let $f : X \to Y$ be continuous. The direct image of sheaves is exactly the direct image of the underlying sheaf, that is,

$$f_* : \mathrm{Sh}_{\mathcal{C}}(X) \to \mathrm{Sh}_{\mathcal{C}}(Y), \quad F \mapsto [f_*F : V_{\in \mathsf{Open}(Y)} \mapsto F(f^{-1}(V))]. \tag{3.2}$$

Remark 3.6. The direct image functor indeed sends sheaves to sheaves, since f^{-1} is an exact functor. The direct image is calculated as presheaves, since f_* is an right adjoint commuting with i.

Definition 3.7 (Inverse image for sheaves)**.** Let $f : X \to Y$ be continuous, where f_p is the inverse image of presheaves. By adjunction, $f^{-1} : \mathrm{Sh}_{\mathcal{C}}(Y) \to \mathrm{Sh}_{\mathcal{C}}(X)$ is defined to be the left adjoint of f_*, i.e.

$$(f^{-1}(-), \cdot)_{\mathrm{Sh}(X)} := ((f_p)^{\#}(-), \cdot)_{\mathrm{Sh}(X)} \simeq (f_p(-), i(\cdot))_{\mathrm{PSh}(X)} \simeq (-, f_* i(\cdot))_{\mathrm{PSh}(Y)} \simeq (-, f_*(\cdot))_{\mathrm{Sh}(Y)} \tag{3.3}$$

Slogan. Stalks to points is what inverse images to open sets. Inverse images are "generalised stalks".

Example 3.8. Recall that,
(1) for $p : \{*\} \hookrightarrow X$, the adjunction $(p^{-1} \dashv p_*)$ refers to stalk and skyscraper sheaf at $p \in X$;

(2) for $\pi : X \twoheadrightarrow \{*\}$, the adjunction $(\pi^{-1} \dashv \pi_*)$ refers to global section and constant sheaf.

Remark 3.9. By definition, constant sheaves are sheafification of constant presheaves, equivalently,
(1) a constant sheaf $\underline{A}$ is exactly the sheaf F with $(F)_x \equiv A$, or
(2) a constant sheaf $\underline{A}$ is exactly the sheaf F with $\Gamma(U, F) = A^{\text{number of connected components of } U}$.

Proposition 3.10 (Exactness of f^{-1})**.** *f^{-1} is an exact functor, since it is defined by purely filtered colimit (f_p-part) and sheafification (($f_p \to f^{-1}$)-part).*
A trick: $(f^{-1}F)_p = p^{-1}f^{-1}F = (f(p))^{-1}F = (F)_{f(p)}$.

Example 3.11. Direct image is not always left exact: just think about the right derivates of global sections.

3.2. Triple from open immersion: $j^! \dashv j^{-1} \dashv j_*$.

Definition 3.12. Say $j : U \to X$ is an open immersion, whenever j factors through an open subset of X via an isomorphism. One can simply view j as an inclusion of open subset when there is no ambiguity.

Example 3.13 ($j_p \dashv j_*$)**.** The direct image $j_* : \mathrm{PSh}_{\mathcal{C}}(U) \to \mathrm{PSh}_{\mathcal{C}}(X)$ sends F to

$$(j_*F) : V \mapsto F(j^{-1}(V)) = F(U \cap V). \tag{3.4}$$

The inverse image $j_p : \mathrm{PSh}_{\mathcal{C}}(X) \to \mathrm{PSh}_{\mathcal{C}}(U)$ sends G to

$$(j_pG) : W \mapsto \varinjlim_{W \subseteq j(V)} G(V) = G(W). \tag{3.5}$$

(1) (Unit) The unit $[\mathrm{id} \to j_*j_p] : \mathrm{PSh}_{\mathcal{C}}(X) \to \mathrm{PSh}_{\mathcal{C}}(X)$ takes F to the restriction $F(- \cap U)$.
(2) (Co-unit) Since j_* is fully faithful, the co-unit $[j_pj_* \to \mathrm{id}] : \mathrm{PSh}_{\mathcal{C}}(U) \to \mathrm{PSh}_{\mathcal{C}}(U)$ is identical.

Example 3.14 ($j^{-1} \dashv j_*$)**.** In the above example, once G is a sheaf, $j_pG = j^{-1}G$ is also a sheaf.

(1) (Direct image) $j_* : \mathrm{Sh}_{\mathcal{C}}(U) \to \mathrm{Sh}_{\mathcal{C}}(X)$ takes $F(-)$ to $F(U \cap -)$.
(2) (Inverse image) $j^{-1} : \mathrm{Sh}_{\mathcal{C}}(X) \to \mathrm{Sh}_{\mathcal{C}}(U)$ takes $G(-)$ to the restriction $G|_U$.
(3) (Unit) $[\mathrm{id} \to j^{-1}j_*] : \mathrm{Sh}_{\mathcal{C}}(U) \to \mathrm{Sh}_{\mathcal{C}}(U)$ is an isomorphism. Hence, j_* is fully faithful.
(4) (Counit) $[j_*j^{-1} \to \mathrm{id}] : \mathrm{Sh}_{\mathcal{C}}(X) \to \mathrm{Sh}_{\mathcal{C}}(X)$ sends G to $G(U \cap -)$.

Notation. If $\iota : Y \subseteq X$ is an inclusion of a subspace, then write $(\cdot)|_Y$ as the inverse image $\iota^{-1}(\cdot)$.

Definition 3.15 (Occasional left adjoint $j_{p!} \dashv j_p$)**.** Suppose that the algebraic structure $\mathcal{C}$ has the initial object $\perp$[3]. For $G \in \mathrm{PSh}_{\mathcal{C}}(X)$ and $F \in \mathrm{PSh}_{\mathcal{C}}(U)$, consider

$$\begin{aligned}(F, j_p G)_{\mathrm{PSh}_{\mathcal{C}}(U)} &\simeq \{F(W) \to G(W)\}_{W \in \mathsf{Open}(U)} && (3.6)\\ &\simeq \{F(W) \to G(W)\}_{W \in \mathsf{Open}(U)} \sqcup \underbrace{\{\perp \to F(W)\}_{W' \notin \mathsf{Open}(X)}}_{\text{useless information}} && (3.7)\\ &=: (j_{p!}F, G)_{\mathrm{PSh}_{\mathcal{C}}(X)}. && (3.8)\end{aligned}$$

Here $j_{p!}(\cdot) : \mathrm{PSh}_{\mathcal{C}}(U) \to \mathrm{PSh}_{\mathcal{C}}(X)$ is the extension by initial.

(1) The unit $\mathrm{id} \to j_p j_{p!}$ is identical (For adjoint triple $F \dashv G \dashv H$, F is fully faithful whenever H is).

(2) The counit $j_{p!} j_p \to \mathrm{id}$ send F to

$$W \to F(U) \quad (U \subseteq W); \qquad W \mapsto \perp \quad (\text{otherwise}). \tag{3.9}$$

Example 3.16. ($j_! \dashv j^{-1}$) For any $F \in \mathrm{Sh}_{\mathcal{C}}(U)$, one has $j_{p!}F \in \mathrm{Sh}_{\mathcal{C}}(X)$. Denote the sheaf functor $j_!$.

(1) (Unit) $\mathrm{id} \to j^{-1} j_! : \mathrm{Sh}_{\mathcal{C}}(U) \to \mathrm{Sh}_{\mathcal{C}}(U)$ is an isomorphism.

(2) (Counit) $j_! j^{-1} \to \mathrm{id} : \mathrm{Sh}_{\mathcal{C}}(X) \to \mathrm{Sh}_{\mathcal{C}}(X)$ send F to $j_!(F|_U)$.

Remark 3.17. On the level of stalks, $(j_!F)_x = F_x$ whenever $x \in U$; otherwise $(j_!F)_x = \perp$.

Slogan. When the domain W has parts outside of U, the set of sections on W is $\perp$.

Proposition 3.18 (Exactness of $j_!$)**.** *On level of stalks, $j_!$ brings more isomorphic stalks.*

(1) *$j_!$ preserves monomorphisms, epimorphisms, and isomorphisms (check in stalk level).*

(2) *$j_!$ is right exact since it is the left adjoint; $j_!$ is not right exact in general (consider $\varprojlim \emptyset = \top$);*

(3) *For Abelian categories, $j_!$ preserves exact sequences, thus is exact.*

[3] The terminal object always exists, since $\varprojlim \emptyset = \top$; whereas the initial object may not exists (e.g. $\mathsf{Sets}_{\neq\emptyset}$)

3.3. Triple from closed immersion: $i^{-1} \dashv i_* \dashv i^!$.

Definition 3.19. Say $i : Z \to X$ is a closed immersion, whenever i factors through an closed subset of X via an isomorphism. One can simply view i as an inclusion of a closed subset when there is no ambiguity.

Example 3.20 ($(i_p \dashv i_*)$)**.** The functors are much more implicit than those in open sets.
The direct image is also a fully faithful functor. The inverse image i_p of presheaf sends $G \in \mathrm{PSh}_{\mathcal{C}}(X)$ to

$$(i_p G)(U \cap Z) = \varinjlim_{i(U \cap Z) \subseteq W} G(W). \tag{3.10}$$

In this sense, $(i_p G)(U \cap Z)$ is the filtered colimit of the section over open sets containing $U \cap Z$.
The unit and co-unit is still implicit at this moment.

Remark 3.21*.* The most special thing about close embedding is that, for any $p \notin Z$, there is some $U_p \cap Z = \emptyset$.

Proposition 3.22. *By examination on level of stalks, i_* preserves monomorphisms, epimorphism and isomorphisms.*
The right adjunction, i_*, is not right exact in general: once the direct image of $i : \emptyset \to X$ preserves finite co-prods, one has

$$\top \simeq i_*(F \coprod G) \simeq i_*(F) \coprod i_*(G) \simeq \top \coprod \top. \tag{3.11}$$

For Abelian sheaves, i_* is exact since it preserves exact sequences.

Remark 3.23*.* There is also an occasionally right adjoint is i_* when $\mathcal{C}$ is "good enough".

Notation. $\mathcal{C}$ is assumed to have a zero object when we discuss supports.

Definition 3.24 (Support of a section)**.** Take $F \in \mathrm{Sh}_{\mathcal{C}}(X)$, and $s \in F(U)$. The support of a section, $\mathrm{Supp}(s)$, is defined by the following equivalent statements.

(1) $\mathrm{Supp}(s) := \overline{\{s(p) \neq 0 \mid p \in U\}}$, where this definition is from analysis.
(2) $\mathrm{Supp}(s) := \{p \mid [s]_p \neq 0\}$, the definition via germs.
(3) $\mathrm{Supp}(s) := \left(\bigcup_{P(U)} U\right)^c$, where $P(U)$ means that $s|_U = 0$ (contrapose the second statement).

Definition 3.25 (Support of a sheaf)**.** Take $F \in \mathrm{Sh}_{\mathcal{C}}(X)$. The support of a sheaf, $\mathrm{Supp}(F) = \{x \mid F_x \neq 0\}$.

Remark 3.26. Supp(F) is not necessary a closed subset, e.g., take $X = \mathbb{R}$ the standard topology, $\mathcal{C} = \mathsf{Mod}_{\mathbb{R}}$, and F the sheaf of continuous functions supported in $\mathbb{R}_{>0}$. Now $\mathrm{Supp}(F) = \mathbb{R}_{>0}$ is not closed.
For some particular kinds of $\mathcal{C}$ (e.g., Ring), $\mathrm{Supp}(\mathcal{O}_X) = \mathrm{Supp}(1_{\in\Gamma(X,\mathcal{O}_X)})$ must be a closed set.

Definition 3.27 (The occasional right adjoint of i_*)**.** For any $F \in \mathrm{Sh}_{\mathcal{C}}(Z)$ and $G \in \mathrm{Sh}_{\mathcal{C}}(X)$,

$$\begin{aligned}(i_*F, G)_{\mathrm{Sh}_{\mathcal{C}}(X)} &\simeq \{(i_*F)(U) \to G(U)\}_{U \in \mathsf{Open}(X)} &(3.12)\\ &\simeq \{F(U\cap Z) \to G(U)\}_{U \in \mathsf{Open}(X)} &(3.13)\\ &\simeq \Big\{F(U\cap Z) \to \underbrace{\{s \in G(U) \mid [s]_p \neq 0 \text{ for any } p \in U \cap Z\}}_{\text{sections supported in } Z}\Big\}_{U \in \mathsf{Open}(X)} &(3.14)\\ &\overset{\star}{=}: (F, i^!(G))_{\mathrm{Sh}_{\mathcal{C}}(Z)}. &(3.15)\end{aligned}$$

Here $i_*i^!(G) \in \mathrm{Sh}_{\mathcal{C}}(X)$ is exactly the kernel of $G(-) \to G(- \cap (Z)^c)$ (check in stalk level), thus is a sheaf.

Remark 3.28. There is no need to discuss $i^!$ for presheaves, since our definition depend on stalks.

Example 3.29. The right adjoint $i^!$ does not necessary preserves epimorphisms. Otherwise, then we take the unit map of the adjunction $\eta_F : F \twoheadrightarrow i_*i^{-1}(F)$, which is epimorphism as one can check on level of stalks. By assumption, the following composition should be an epimorphism

$$i^!(F) \xrightarrow{i^!(\eta_F)} i^!(i_*i^{-1}F) \xrightarrow[\sim]{i_* \text{ is fully faithful}} i^{-1}(F). \tag{3.16}$$

Now consider the closed embedding $p : \{*\} \hookrightarrow \mathbb{R}$. For $F \in \mathrm{Sh}_{\mathcal{C}}(\mathbb{R})$ the above morphism is $0 \to F_p \neq 0$.

3.4. **Topological Six Functors on Presheaves.**

Notation. Let $i : Z \to X$ is a closed immersion and $j : U \to X$ is an open immersion. $Z = U^c$.

Example 3.30. For $\mathcal{C}$ with initial object $\perp$, we have the diagram of five functors:

$$\mathrm{Sh}_{\mathcal{C}}(U) \;\; \begin{matrix} \xrightarrow{\quad j_! \quad} \\ \xleftarrow{\quad j^{-1} \quad} \\ \xrightarrow[\quad j_* \quad]{} \end{matrix} \;\; \mathrm{Sh}_{\mathcal{C}}(X) \;\; \begin{matrix} \xrightarrow{\quad i^{-1} \quad} \\ \xleftarrow{\quad i_* \quad} \end{matrix} \;\; \mathrm{Sh}_{\mathcal{C}}(Z) \ . \tag{3.17}$$

Here $j_! \dashv j^{-1} \dashv j_*$ is the adjoint triple, where $j_!$ and j_* are fully faithful.
The composition of the first row, $i^{-1} \circ j_!$ takes $H \in \mathrm{Sh}_{\mathcal{C}}(U)$ to the constant sheaf $\underline{\perp} \in \mathrm{Sh}_{\mathcal{C}}(U)$ [4].
The composition of the second row sends $G \in \mathrm{Sh}_{\mathcal{C}}(Z)$ to the constant sheaf $\underline{\top} \in \mathrm{Sh}_{\mathcal{C}}(U)$.
For $F \in \mathrm{Sh}_{\mathcal{C}}(X)$, the sequence

$$[j_! j^{-1} F \xrightarrow{\text{counit}} F \xrightarrow{\text{unit}} i_* i^{-1} F] \quad = \begin{cases} [F_p = F_p \to \top] & p \in U, \\ [\perp \to F_p = F_p] & p \in Z. \end{cases} \tag{3.18}$$

Example 3.31. Suppose the category has zero object, we have the full diagram of 6 functors:

$$\mathrm{Sh}_{\mathcal{C}}(U) \;\; \begin{matrix} \xrightarrow{\quad j_! \quad} \\ \xleftarrow{\quad j^{-1} \quad} \\ \xrightarrow[\quad j_* \quad]{} \end{matrix} \;\; \mathrm{Sh}_{\mathcal{C}}(X) \;\; \begin{matrix} \xrightarrow{\quad i^{-1} \quad} \\ \xleftarrow{\quad i_* \quad} \\ \xrightarrow[\quad i^! \quad]{} \end{matrix} \;\; \mathrm{Sh}_{\mathcal{C}}(Z) \ . \tag{3.19}$$

Here the left and right parts are adjoint triple, the composition of each row is 0.

- (Explanation of $i^! j_* \simeq 0$.) For any $F \in \mathrm{Sh}_{\mathcal{C}}(U)$, any section of $(j_* F)(W)$ supported in $Z \cap W$ is zero.

[4] The sheaf takes value $\top$ for on $\emptyset$ and $\perp$ on non-empty sets. Since the product of $\perp$ is $\perp$ itself.

Since $\top = \bot = 0$, the sequence completes to

$$0 \to j_! j^{-1} F \to F \to i_* i^{-1} F \to 0. \tag{3.20}$$

There is also a functorial kernel sequences

$$0 \to i_* i^! F \to F \to j_* j^{-1} F. \tag{3.21}$$

Clearly, $i_* i^! F \to F$ is a monomorphism (injection). For exactness[5] at F:

(1) Assume $p \in Z$. For any $[s]_p \in F_p$, whenever there exists $t \in [s]_p$ supported in Z, t has zero germs in V.
(2) Assume $p \in U$. Trivial.

Proposition 3.32. *Let $\mathcal{C}$ be an Abelian category.* $\mathrm{PSh}_{\mathcal{C}}(\cdot)$ *is also Abelian.*

$$\mathrm{Sh}_{\mathcal{C}}(U) \; \underset{j_*}{\overset{j_!}{\rightrightarrows}} \; \overset{j^{-1}}{\longleftarrow} \; \mathrm{Sh}_{\mathcal{C}}(X) \; \underset{i^!}{\overset{i^{-1}}{\rightrightarrows}} \; \overset{i_*}{\longleftarrow} \; \mathrm{Sh}_{\mathcal{C}}(Z) \; . \tag{3.22}$$

Red (resp. blue, purple) arrows are exact (resp. fully faithful, both). Moreover,

$$0 \to j_! j^{-1} F \to F \to i_* i^{-1} F \to 0, \tag{3.23}$$

$$0 \to i_* i^! F \to F \to j_* j^{-1} F \qquad (\textit{not right exact}). \tag{3.24}$$

Remark 3.33. The second arrow is not right exact in general. The first sequence is deduced purely on stalks level; whereas the deduction of the second sequence is partially based on open sets.

Definition 3.34 (Flasque sheaf)**.** Say $F \in \mathrm{Sh}(F)$ is flasque, whenever Res is always a surjection.

[5] Let $\mathcal{C}$ be with zero object. Say $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact, whenever $g^{-1}(0) = \mathrm{im}(f)$.

Remark 3.35. For flasque sheaf F, the morphism $F \to j_*j^{-1}F$ is a surjection and thus an epimorphism. The second sequence completes to a short exact sequence

$$0 \to i_*i^!F \to F \to j_*j^{-1}F \to 0. \tag{3.25}$$

Example 3.36. Roughly speaking, the global section of a flasque sheaf carries information of section on the subspaces. The flasque sheaf is somehow related to "separable". For instance,

(1) The constant sheaves are not flasque in general, e.g. $F(\mathbb{R}) \to F(\mathbb{R}_\pm) \simeq F(\mathbb{R})^{\oplus 2}$.

(2) $(\Pi F)(\cdot) := \prod_{p\in(\cdot)} F_p$ is flasque. Hence every sheaf embeds to some flasque sheaves.

(3) One can apply "Yoneda trick" to some of the categories, e.g. for any inclusion $i : U \to V$, one has

$$F(i) = (F(V) \to F(U))_{\mathsf{Set}} \simeq ((h_V, F)_{\mathsf{PSh}(X)} \to (h_U, F)_{\mathsf{PSh}(X)}) = (h_i, F)_{\mathsf{PSh}(X)}. \tag{3.26}$$

Here $h_U := (-, U)_{\mathsf{Open}(X)}$. If (h_i, F) is an surjection (e.g., F is injective), then $F(i)$ is an surjection.

Proposition 3.37. *For a cateogorical explanation of "flasque", the right derive $\mathrm{R}^\bullet\Gamma(U, -)$ vanishes.*

Take $0 \to F \xrightarrow{f} G \xrightarrow{g} H \to 0$ as the exact sequence for sheaves where F is flasque. We shall show that $G(U) \xrightarrow{f_U} H(U)$ is always a surjection.

For any section $h \in H(U)$, the test of sheaf epimorphism says that

- *there exists a gluing $h = \bigsqcup_I h_i/\sim$ such that every $h_i \in H(U_i)$ is the image of some g_i.*

For any pair (i, j), one has

$$(g_i - g_j)|_{U_i\cap U_j} \in \ker(g_{U_i\cap U_j}) = \mathrm{im}(f_{U_i\cap U_j}). \tag{3.27}$$

Now we can free extend the sections in F! We assume F is a subsheaf of G for a shorthand. Take

$$F(U_i) \twoheadrightarrow F(U_i \cap U_j), \quad g^i_{i,j} \mapsto (g_i - g_j), \tag{3.28}$$

such that, by gluing $g^i_{i,j}\ g^j_{j,i}$, one has $g_{i,j} \in F(U_i \cup U_j)$. By such procedure, one can "kill" any fintie many indices in I. The "exhaustion" of the induction gives $h \in \mathrm{im}(f_U)$, which shows that $G(U) \to H(U)$ is a surjection. The "exhaustion" is reasonable, since filtered colimits preserves open sets, and I is a filtered colimits of its finite subsets.

Remark 3.38. The proof is similar for pointed categories.

Proposition 3.39 (Partially 2-out-of-3)**.** *Take short exact sequence of sheaves, $0 \to F \xrightarrow{f} G \xrightarrow{g} H \to 0$. Suppose that F is flasque, then G is flasque whenever H is flasque.*
For any inclusion $\iota : U \subseteq V$, one has $\mathrm{coker}(F(\iota)) = 0$. Hence $\mathrm{coker}(G(\iota)) = 0 \iff \mathrm{coker}(H(\iota)) = 0$.

Remark 3.40. Left adjoint functors preserves flasqueness, by commuting lemmas.

Example 3.41 (MV base sequence)**.** We rewrite ($\Rightarrow$ occasionally exists)

$$0 \to \underbrace{\Gamma_Z(F)}_{i_* i^! F} \to F \to \underbrace{\Gamma_U(F)}_{j_* j^{-1}(F)} \Rightarrow 0; \qquad 0 \to \underbrace{F_U}_{j_! j^{-1} F} \to F \to \underbrace{F_Z}_{i_* i^{-1} F} \to 0. \tag{3.29}$$

By checking exactness on level of stalks, one has the following sequences.

(1) For covering of closed subspaces $Z_1 \cup Z_2 = X$, there is a pullback-pushout square:

$$0 \to F \to F_{Z_1} \oplus F_{Z_2} \to F_{Z_1 \cap Z_2} \to 0. \tag{3.30}$$

(2) For covering of closed subspaces $Z_1 \cup Z_2 = X$, there is a pullback square:

$$0 \to \Gamma_{Z_1 \cap Z_2}(F) \to \Gamma_{Z_1}(F) \oplus \Gamma_{Z_2}(F) \to F \Rightarrow 0. \tag{3.31}$$

(3) For open covering of open subspaces $U_1 \cup U_2 = X$, there is a pullback square

$$0 \to F_{U_1 \cap U_2} \to F_{U_1} \oplus F_{U_2} \to F \to 0. \tag{3.32}$$

(4) For open covering of open subspaces $U_1 \cup U_2 = X$, there is a pullback square

$$0 \to F \to \Gamma_{U_1} \oplus \Gamma_{U_2} \to \Gamma_{U_1 \cap U_2} \Rightarrow 0. \tag{3.33}$$

Example 3.42 (Example from Homological Algebra)**.** The ideal comes from quasi-coherency, roughly speaking, finitely presented modules.
Let $\mathcal{C}$ be an additive category.

Example 3.43 (More on constant presheaf)**.** Take the canonical morphism $\pi : X \twoheadrightarrow \{*\}$.

(1) The direct image $\pi_* : \mathrm{PSh}_{\mathcal{C}}(X) \to \mathcal{C}$ is just the global section.
(2) The inverse image $\pi_p : \mathcal{C} \to \mathrm{PSh}_{\mathcal{C}}(X)$ sends A to the constant presheaf $\underline{A}$.
(3) We claim that $\pi_{p!} : \mathrm{PSh}_{\mathcal{C}}(X) \to \mathcal{C}$ sends F to $F(\emptyset)$, since

$$(\pi_{p!} \dashv \pi_p \dashv \pi_*) \iff (\varinjlim_{\mathsf{Open}(X)^{\mathrm{op}}} \dashv \underline{(-)} \dashv \varprojlim_{\mathsf{Open}(X)^{\mathrm{op}}}) \tag{3.34}$$

(4) "$\pi_{p!}$" has a left adjoint, constructed by "void map $i : \emptyset \to X$". We find that $\pi_{p!} = i^{-1} \vdash i_{p!}$, sending

$$\mathcal{C} \to \mathrm{PSh}_{\mathcal{C}}, \quad A \mapsto [\emptyset \mapsto A, \quad (\neg\,\emptyset) \mapsto \bot]. \tag{3.35}$$

Form the perspective of free-forgetful adjunction, $\pi_{p!}$ forgets only properties since it is the left-left adjoint of a fully faithful functor. The "free" left adjoint of $\pi_{p!}$ restrict anything from the initial.

(5) π_* also has a right adjoint. By observation, $\varinjlim_{\mathsf{Open}(X)^{\mathrm{op}}}$ purely depends on X, i.e. independent from the choice of topology. It is straight forward to verify the right adjoint of π_* sends

$$\mathcal{C} \to \mathrm{PSh}_{\mathcal{C}}(X), \quad A \mapsto [X \to A, \quad \neg\, X \mapsto \top]. \tag{3.36}$$

It follows that we have the adjoint pentuple

$$A \mapsto \begin{bmatrix} X \mapsto A \\ \neg X \mapsto \top \end{bmatrix} \quad \dashv \quad \Gamma(\emptyset, -) \quad \dashv \quad \underline{(-)} \quad \dashv \quad \Gamma(X, -) \quad \dashv \quad A \mapsto \begin{bmatrix} \neg \mapsto \top \\ \emptyset \mapsto A \end{bmatrix}. \tag{3.37}$$

Remark 3.44. Consider $\to \sim \ \mathsf{Open}(\{*\})$. One has $(\bot \to ?) \dashv t \dashv \mathrm{id}_{\bullet} \dashv s \dashv (? \to \top)$ for morphism categories.

Example 3.45. A basic fact: the adjoint triple $L \dashv M \dashv R$ gives adjoint endofunctors $LM \dashv RM$.
For instance, let $\iota : \mathsf{Ord}_{\leq n} \to \mathsf{Ord}_{<\omega}$ be the inclusion of finite ordinals $\leq n$.
One can identify $\iota = \pi^{-1} := \mathsf{Open}([n]) \to \mathsf{Open}(\mathbb{N})$ where $\pi(d) = \min(d, n)$. Hence,

$$[\pi_{p!} \dashv \pi_p \dashv \pi_*] \Rightarrow [(\pi_{p!}\pi_p) \dashv (\pi_*\pi_p)]. \tag{3.38}$$

It is exactly the adjoint of "skeleton" $\dashv$ "coskeleton".

4. Recollement of categories

4.1. Six functors from Localisations of Additive Categories.

Notation. For the most basic settlement for homological algebra, categories are additive.

Example 4.1 (Where to glue $\mathcal{C}$?)**.** For $\mathcal{C}$ be additive. Take the Yoneda embedding

$$\iota_* : \mathcal{C} \to \mathrm{AddFunct}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}), \quad \star \mapsto (-, \star). \tag{4.1}$$

We write $\mathrm{AddFunct}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) =: \mathrm{Ab}(\mathcal{C})$ for simplicity. Now there is

$$? \dashrightarrow \mathrm{Ab}(\mathcal{C}) \overset{?}{\underset{j_*}{\rightleftarrows}} \mathcal{C} \ . \tag{4.2}$$

Remark 4.2. $\mathrm{Ab}(\mathcal{C})$ is somehow too large to handle. In practice, we usually consider the "smallest" Abelian category which has a subcategory equivalent to $\mathcal{C}$.
In short, the aim is to find $\mathcal{C} \hookrightarrow \mathcal{C}$ where the least information is lost.

Example 4.3. Here is a sequence where we make an additive category $\mathcal{C}$ "complete".

$$\begin{array}{ccccccccc} \text{additive} & \longrightarrow & \text{Karoubian} & \longrightarrow & \text{finite cocomplete} & \longrightarrow & \text{finite cosmoi} & \longrightarrow & \text{cosmoi} \\ \mathcal{C} & \longrightarrow & \overline{\mathcal{C}}^{\oplus} & \longrightarrow & \overline{\mathcal{C}}^{\text{coker}} & \longrightarrow & \text{ab}(\mathcal{C}) & \longrightarrow & \text{Ab}(\mathcal{C}) \end{array} \quad . \tag{4.3}$$

(1) Kabourian means that idempotent morphism $f^2 = f$ has kernel and cokernels, or simply complete w.r.t. summand for additive cases.

Example: let $\mathcal{C}$ be the category wherein the objects are small open subsets of $\bigcup_{n<\omega} \mathbb{R}^n$, and moprhisms are smooth maps. Now $\overline{\mathcal{C}}^{\oplus}$ is the category of smooth manifolds.

(2) Finite cocompleteness means all small diagram has colimits, or simply "cokernels exists" for additive cases.

(3) Cosmoi means complete and cocomplete.

Proposition 4.4. $\overline{\mathcal{C}}^{\text{coker}}$ *quotient of the morphism category* $\mathcal{C}^{\rightarrow}$*.*
Notice that $\overline{\mathcal{C}}^{\text{coker}}$ *is taken from the Yoneda embedding* $\mathcal{C} \hookrightarrow \text{Ab}(\mathcal{C})$*. Hence,*

- $F \in \mathsf{Ob}(\overline{\mathcal{C}}^{\text{coker}})$ *is represented by some* $X \xrightarrow{f} Y \in \mathsf{Mor}(\mathcal{C})$*, via* $(-, X) \xrightarrow{(-,f)} (-, Y) \to F(-) \to 0$*.*

To see that $\overline{\mathcal{C}}^{\text{coker}}$ *has cokernels, we take arbitrary* $F_i := \text{coker}((-, f_i))$ *and consider the* 2*-term projective resolution* $P_i^{\bullet}$*.* $\text{coker}(\varphi : F_1 \to F_2)$ *is exactly* $H^0(\text{Cone}(\varphi^{\bullet})) \simeq \text{coker}((-, P_2^1 \oplus P_1^0) \to (-, P_2^0))$*.*

Definition 4.5 (Finite presented)**.** For additive cateogry $\mathcal{C}$ coker, define the finitely presented category $\text{ab}(\mathcal{C})$ as the homotopic category $\mathcal{C}^{\rightarrow}/\sim$. That in, the objects are of the form $F(-)$, identified by a morphism

$$(-, X) \xrightarrow{(-,f))} (-, Y) \to F(-) \to 0. \tag{4.4}$$

Proposition 4.6. *$\overline{\mathcal{C}}^{\text{coker}}$ is abelian, iff $\overline{\mathcal{C}}^{\text{coker}}$ has kernels, iff ≥ 3 terms projective resolution exists, iff any $(-, Y) \to (-, Z)$ fits into an exact sequence $(-, X) \to (-, Y) \to (-, Z)$, iff $\mathcal{C}$ has weak kernel.*

Remark 4.7. By Yoneda trick, $(-, X)$ is the projective object in $\overline{\mathcal{C}}^{\text{coker}}$. Here is a comparision with Mod:

- $(-, X)$ is to finitely genereated projective modules, is what $F \in \overline{\mathcal{C}}^{\text{coker}}$ to finitely presented modules.

Example 4.8. Now, our diagram becomes

$$\begin{array}{ccccc} & & \operatorname{coker}(-, \varphi) & \xrightarrow{\ ?\ } & ? \\ \ker(i^{-1}) & \xrightarrow{\ i_*\ } & \overline{\mathcal{C}}^{\text{coker}} & \xrightarrow{\ j^{-1}\ } & \mathcal{C} \\ & & & \xleftarrow{\ j_*\ } & \\ & & (-, X) & \longleftarrow & X \end{array} \tag{4.5}$$

Where the red arrows are fully faithful. Whenever j^{-1} exists, it follows from the adjunction that

$$\begin{aligned} (j^{-1}(\operatorname{coker}(-, \varphi)), X)_{\mathcal{C}} &\simeq (\operatorname{coker}(-, \varphi), (-, X))_{\operatorname{coker}(\mathcal{C})} && (4.6) \\ &\simeq \ker((-, \varphi), (-, X))_{\operatorname{coker}(\mathcal{C})} && (4.7) \\ \text{(Yoneda)} \qquad &\simeq \ker(\varphi, X)_{\mathcal{C}} && (4.8) \\ &\simeq (\operatorname{coker}(\varphi), X)_{\mathcal{C}}. && (4.9) \end{aligned}$$

Remark 4.9. It is resonable to assume $\mathcal{C}$ has cokernels.

Proposition 4.10. *We have show* $0 \to i_! i^! F \to F \to j_* j^{-1} F$ *for sheaves, thus wish that*

$$\begin{array}{ccccc} & & \operatorname{coker}(-,\varphi) & \longmapsto & \operatorname{coker}(\varphi) \\ \ker(i^{-1}) & \underset{\dashleftarrow}{\xrightarrow{i_*}} & \overline{\mathcal{C}}^{\operatorname{coker}} & \underset{\xleftarrow[j_*]{}}{\xrightarrow{j^{-1}}} & \mathcal{C} \\ \ker[\operatorname{coker}(-,\varphi) \to (-,\operatorname{coker}\varphi)] & \overset{i^!}{\longleftarrow\!\shortmid} & \operatorname{coker}(-,\varphi) & & \end{array} \quad . \tag{4.10}$$

is an adjunction. To see that $(G(-), j^{-1}(\operatorname{coker}(-,\varphi))) \simeq (G(-), (-, \operatorname{coker}(\varphi)))$*, it suffices to show that any compositiion*

$$G(-) \to \operatorname{coker}(-,\varphi) \to (-, \operatorname{coker}(\varphi)) \quad (\varphi : A \to B) \tag{4.11}$$

is zero. For any $X \in \mathcal{C}$*, the map* $G(X) \to (X, \operatorname{coker}(\varphi))$ *factors through*

$$(X, A) \to (X, B) \to (X, \operatorname{coker}(\varphi)), \tag{4.12}$$

which is zero.

Remark 4.11. Form the perspective of stable categories, $j^{-1} : \overline{\mathcal{C}}^{\operatorname{coker}} \to \mathcal{C}$ erases phantom functors.

Definition 4.12 (Localisation of categories)**.** The general approach for localisation: given any category $\mathcal{C}$, and a class of morphisms S, construct the "path cateogory" by adding inverse morphisms in S, define some resonable equivalent classes of morphisms, and finally define the quotient category $\mathcal{C}[S^{-1}]$. The localisation is the composition of

$$Q : \mathcal{C} \xrightarrow{\text{generate the path category}} \widetilde{\mathcal{C}} \xrightarrow{\text{quotient some paths}} \mathcal{C}[S^{-1}], \tag{4.13}$$

which in identical for objects, and morphisms are equivalent classes of the compositon

$$\to \circ \Longleftarrow \circ \to \circ \Longleftarrow \circ \to \circ \Longleftarrow \circ \underbrace{\cdots}_{\text{finite many}} \to \circ \Longleftarrow \circ \to \circ \Longleftarrow . \tag{4.14}$$

Here $\Longleftarrow$ is the formal inverse of some S in $\widetilde{\mathcal{C}}$.
We highlight that the construction is based on NGBC axiom system rather than ZFC.

Remark 4.13. The "equivalent classes in a category" is the same thing as the filtered colimits of large diagram (whose base set is a proper class). It is no need to introduce NGBC axioms strictly; just remember that, never take class of classes, and always state the uniqueness under equivalences.

Proposition 4.14 (Universal property of localisation)**.** *Let $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ be the localisation. For any functor $F : \mathcal{C} \to \mathcal{D}$ sending each $s \in S$ to some isomoprhism in $\mathsf{Mor}(\mathcal{D})$, there is a factorisation*

$$\mathcal{C} \xrightarrow{Q} \mathcal{C}[S^{-1}] \xrightarrow{\overline{F}} \mathcal{D}. \tag{4.15}$$

In sense of equivalences of categories, the factorisation is unique.

Remark 4.15. This is not the "universal property" in our convention.

Proposition 4.16. *Here are some remarkable properties of localisation which are easy to proof:*
(1) *if $\mathcal{C}[S^{-1}]$ and $\mathcal{C}[T^{-1}]$ factors through each other, then the categories are equivalent;*
(2) *the pre-composition $(- \circ Q)$:* Funct$(\mathcal{C}, \mathcal{D})$ *is fully faithful, i.e. the natural transformations (F_1, F_2) and (F_1Q, F_2Q) coincides;*
(3) *localisation preserves finite colimits*[6]*;*
There are some special cases of localisation for additive categories.
(1) *When S is cloesd under finite "matrix direct sums", then the localisation category is additive.*

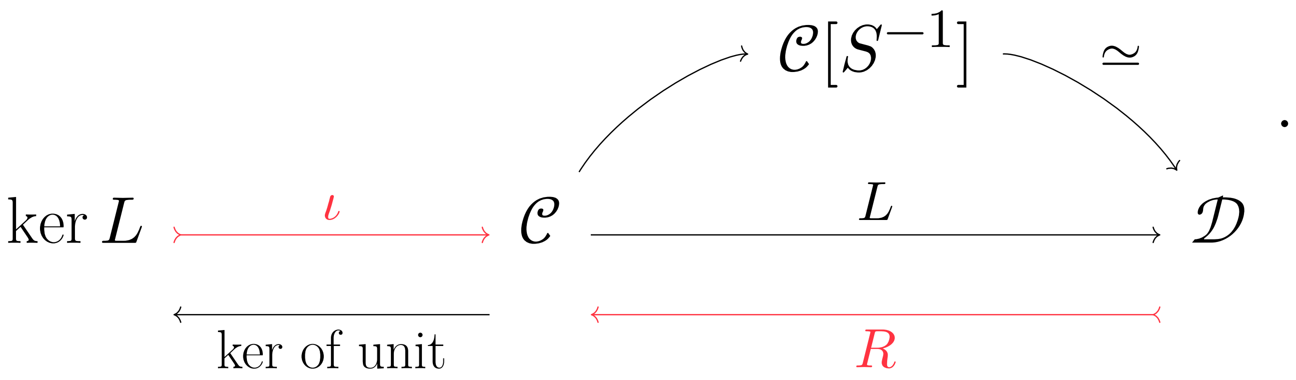
[6] Be careful when during the proof of $(\mathcal{C} \times \mathcal{D})[(S \times T)^{-1}] \simeq (\mathcal{C}[S^{-1}]) \times (\mathcal{D})[T^{-1}]$, one can use "Yoneda lemma" to show that both sides satisfies the same universal property

(2) *The stable category $\mathcal{C}/\mathcal{B}$ is a localisation, where S are morphisms factors through objects in $\mathcal{B}$.*
(3) *For some cases, S satisfies some "Ore properties" such that the zigzag morphisms are simplified.*

The localisation of additive category is not necessary additive: take $\mathsf{Vec}_{\mathbb{R}}$ *and* $S = \{0 \to \mathbb{R}\}$.

Slogan. The upper part of the six functor diagram comes from localisations!

Example 4.17 (The construction of localisation sequences)**.** Let $L : \mathcal{C} \to \mathcal{D}$ be a left adjoint functor, and denote its the right adjoint R. Suppose that there is some class of morphism $S \subseteq \mathsf{Mor}(\mathcal{C})$ such that $L(S)$ are isomorphisms in $\mathcal{D}$.
It takes some times to verify that, R is fully faithful whenever $\mathcal{C}[S^{-1}] \to \mathcal{D}$ is an equivalence. Now,



(4.16)

The proofs on localisations are usually locally complicated and globally unreadable.

If $\overline{L}$ is an equivalence, then we have the following fully faithful pullback

$$L^* : (\mathcal{D}, \mathcal{D}) \to (\mathcal{C}, \mathcal{D}), \quad T \mapsto T \circ L.. \tag{4.17}$$

Hence, L^* gives the isomprhism $\Phi : \mathsf{Nat}[\mathrm{id}_{\mathcal{D}}, LR] \cong \mathsf{Nat}[L, LRL]$. The preimage of $L\eta$ is $\theta : \mathrm{id}_{\mathcal{D}} \to LR$.

(1) The composition of $\mathrm{id}_{\mathcal{D}} \xrightarrow{(\mathrm{id}_{\mathcal{D}})\theta} (\mathrm{id}_{\mathcal{D}})(LR)(\mathrm{id}_{\mathcal{D}}) \xrightarrow{\varepsilon(\mathrm{id}_{\mathcal{D}})} \mathrm{id}_{\mathcal{D}}$ is $\mathrm{id}_{\mathrm{id}_{\mathcal{D}}}$, since

$$\varepsilon\theta \in \quad \mathsf{Nat}[\mathrm{id}_{\mathcal{D}}, \mathrm{id}_{\mathcal{D}}] \cong \mathsf{Nat}[L, L] \quad \ni \varepsilon\theta L. \tag{4.18}$$

Hence, $\left[L \xrightarrow{\varepsilon\theta L} L\right] = \left[L \xrightarrow{\theta L} LRL \xrightarrow{\varepsilon L} L\right] = \left[L \xrightarrow{L\eta} LRL \xrightarrow{\varepsilon L} L\right] = \mathrm{id}_L$.

(2) the composiiton of $LR \xrightarrow{\theta(LR)} (LR)(\mathrm{id}_{\mathcal{D}})(LR) \xrightarrow{(LR)\varepsilon} LR$ is id_{LR}, since

$$(LR)\varepsilon \circ \theta(LR) \in \quad \mathsf{Nat}[LR, LR] \cong \mathsf{Nat}[LRL, LRL] \quad \ni (-)L. \tag{4.19}$$

Now $\left[LRL \xrightarrow{(-)L} LRL\right] = \left[LRL \xrightarrow{\theta LRL} LRLRL \xrightarrow{LR\varepsilon L} LRL\right] = \left[LRL \xrightarrow{L(\eta R)L} LRL \xrightarrow{L(R\varepsilon)L} LRL\right] = \mathrm{id}_{LRL}$.

We learn from above that (id, LR) is a adjoint to $\mathcal{D} \to \mathcal{D}$, hence LR is a natural isomoprhism. As a result, R is fully faithful.

Proposition 4.18 (Construct localisation without "$\mathcal{D}$"). *Let $\mathcal{C}$ be an category. Suppose that there is an endofunctor $I : \mathcal{C} \to \mathcal{C}$ and a natural transformation $\eta_{\mathrm{id}_\mathcal{C}} \to I$, such that*

$$I\eta = \eta I : I \to I \circ I \tag{4.20}$$

is a natural isomorphism. There is a diagram picture for this subsection

$$\begin{array}{ccccc} & & X & \longmapsto & LX \\ \ker L & \xrightarrow{\;i_*\;} & \mathcal{C} & \xrightarrow{\;j^{-1}\;} & \mathrm{im}^1(I) \\ & \xleftarrow[\;i^!\;]{} & & \xleftarrow[\;j_*\;]{} & \updownarrow \simeq \\ \ker(\eta_X) & \longleftarrow & X & & S^\perp \end{array} \; . \tag{4.21}$$

The notation $S^\perp$ is discused as follows.

Example 4.19. There are another ways to charactarise im^1 by taking "sub" instead of the "quotient". Let $S^\perp \subseteq \mathsf{Ob}(\mathcal{C})$ be a collection of objects charactarised by the following equivalent statements.
(1) $X \in S^\perp$ whenever $(-, X)_\mathcal{C} \to (-, X)_{\mathcal{C}[S^{-1}]}$ is an natural isomorphism in $\mathrm{Funct}(\mathcal{C}^{\mathrm{op}}, \mathsf{Sets})$.
(2) $X \in S^\perp$ whenever $(s, X)_\mathcal{C}$ is a bijection for any $s \in S$.
In this case, $S^\perp$ is eqivalent to $\mathrm{im}^1(I)$.

Remark 4.20. The notation and the general theory comes from torsion pairs.

4.2. **Glueing Serre subcategory.**

Definition 4.21. Let $\mathcal{A}$ be an Abelian category. Say the full subcategory $\mathcal{A}^{'}$ is a Serre-subcategory, whenever $\mathcal{A}^{'}$ is closed under subobjects, quotients objects, and extensions.

Slogan. In short, the Serre-subcategory means"two out of three on short exact sequences".

Definition 4.22 (Serre quotient)**.** Let $\mathcal{T} \subseteq \mathcal{A}$ be a Serre sub-category. Define

$$\mathcal{A}/\mathcal{T} := \mathcal{A}[S^{-1}] \quad (s \in S) \iff \ker s \text{ and } \operatorname{coker}(s) \in \mathsf{Ob}(\mathcal{T}). \tag{4.22}$$

Here S is indeed a multiplicative system (both left and right).

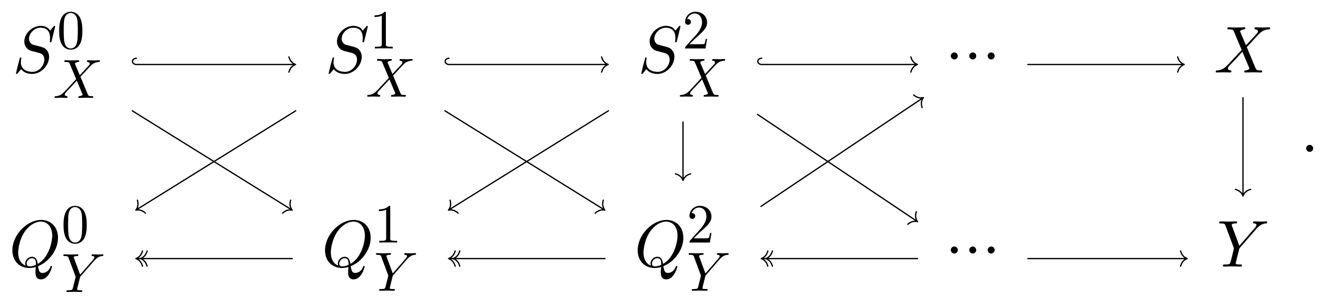
Example 4.23. The construction of $\mathcal{A}/\mathcal{T}$ is implicit. At least it is clueless to determine whether $\mathcal{A}/\mathcal{T}$ is Abelian or not. Recall that the localisation w.r.t. (left) multiplicative system is defined over a "large filtered colimit" (equivalence relation of categories), i.e.

$$\varinjlim_{(s,M)\in I} (X,M)_{\mathcal{A}} \cong (X,Y)_{\mathcal{A}/\mathcal{T}} \quad (I := \{(s,M) \mid (s : Y \to M) \in S\}). \tag{4.23}$$

Since S is closed under epi-mono factorisation, it suffices to see the cofinal filtered system

$$\varinjlim_{(s,M)\in I} (X,M)_{\mathcal{A}} \cong (X,Y)_{\mathcal{A}/\mathcal{T}} \quad (I := \{(s,M) \mid (s : Y \twoheadrightarrow M) \in S\}). \tag{4.24}$$

For right multiplicative system, the filtered colimits takes over subobjects of X.



(4.25)

Suppose that $\mathcal{A}$ somehow concrete (well-powered and AB5), one has

$$(X, Y)_{\mathcal{A}/\mathcal{T}} \simeq \varinjlim (S_X, Q_Y)_{\mathcal{A}} \qquad (S_X \hookrightarrow X \text{ and } Y \twoheadrightarrow Q_Y \in S). \tag{4.26}$$

Now $\mathcal{A}/\mathcal{T}$ is Abelian. For instance, $\ker^{\mathcal{A}/\mathcal{T}}(f \cdot 1^{-1})$ and $\ker^{\mathcal{A}}(f)$ satisfies the same universal property:

$$\{(X, \ker f)_{\mathcal{A}/\mathcal{T}}\}_X \simeq \{\varinjlim_{S_X}(S_X, \ker f)_{\mathcal{A}/\mathcal{T}}\}_X \simeq \{\ker \varinjlim_{S_X}(S_X, f)_{\mathcal{A}/\mathcal{T}}\}_X \simeq \{\ker(X, f)_{\mathcal{A}/\mathcal{T}}\}_X. \tag{4.27}$$

Example 4.24. Some properties related to isomorphism theorems. Here is a comparison with Verdier quotient of triangulated categories.

(1) Kernel of an exact functor is a Serre subcategory.
(2) Kernel of an $\triangle$ functor is thick subcategory.
(3) Whenever exact functor $F : \mathcal{A} \to \mathcal{B}$ annihilates $\mathcal{T}$, F factors through the localisation $\mathcal{A}/\mathcal{T}$.
(4) Whenever $\triangle$ functor $F : \mathcal{A} \to \mathcal{B}$ annihilates $\mathcal{T}$, F factors through the localisation $\mathcal{A}/\mathcal{T}$.
(5) Let $\mathcal{T} \subseteq \mathcal{T}' \subseteq \mathcal{A}$ be 2 inclusions of Serre subcategories, then $\mathcal{T}$ is also the Serre subcategory of $\mathcal{A}$. Moreover, $\mathcal{T}'/\mathcal{T} \subseteq \mathcal{A}/\mathcal{T}$ is also an inclusion of Serre subcategory and $\frac{\mathcal{T}'/\mathcal{T}}{\mathcal{A}/\mathcal{T}} \simeq \mathcal{T}'/\mathcal{A}$.
(6) Let $\mathcal{T} \subseteq \mathcal{T}' \subseteq \mathcal{A}$ be 2 inclusions of thick subcategories, then $\mathcal{T}$ is also the thick subcategory of $\mathcal{A}$. Moreover, $\mathcal{T}'/\mathcal{T} \subseteq \mathcal{A}/\mathcal{T}$ is also an inclusion of thick subcategory and $\frac{\mathcal{T}'/\mathcal{T}}{\mathcal{A}/\mathcal{T}} \simeq \mathcal{T}'/\mathcal{A}$.

Example 4.25. Here we setup of localisation of Serre subcategory. Suppose the exact functor $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ admits two-sided adjoints. Now there is a picture

$$\mathcal{T} \underset{i^!}{\overset{i^{-1}}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\ i_*\ }}}}} \mathcal{A} \underset{j_*}{\overset{j_!}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\ j^{-1}\ }}}}} \mathcal{A}/\mathcal{T} \ . \tag{4.28}$$

Along with the kernel sequence (lower part)

$$0 \to i_* i^! A \to A \to j_* j^{-1} A, \tag{4.29}$$

and the cokernel sequence

$$j_! j^{-1} A \to A \to i_* i^{-1} A \to 0. \tag{4.30}$$

Proposition 4.26. *A criterion for $S^\perp$. For any $Y \in S^\perp$ and any $X \in \mathcal{T}$:*

(1) *$(0_{0X}, Y)_{\mathcal{A}}$ is a bijection, hence $(X, Y)_{\mathcal{A}} = 0$;*

(2) *For any short exact sequence $0 \to Y \xrightarrow{f} E \to X \to 0$, (f, Y) is a bijection. Hence, f is a split monomorphism and thus $\mathrm{Ext}^1(X, Y) = 0$.*

Conversely, suppose that ${}^\perp\mathrm{Hom}(-, Y)$ and ${}^\perp\mathrm{Ext}^1(-, Y)$ are zeros on $\mathcal{C}$.

(1) *For any monomorphism $i \in S$, one has*

$$0 \to A \xrightarrow{i} B \to X \to 0 \implies 0 = (X, Y) \to (B, Y) \xrightarrow{\sim} (A, Y) \to \mathrm{Ext}^1(X, Y); \tag{4.31}$$

(2) *For any epimorphism $p \in S$, one has*

$$0 \to X \to C \xrightarrow{p} D \to 0 \implies 0 \to (D, Y) \xrightarrow{\sim} (C, Y) \to (X, Y) = 0. \tag{4.32}$$

Hence, $Y \in S^\perp$.

Remark 4.27. The analogue in triangulated is the t-structure.

Example 4.28. Where the ideals comes from? Let $\mathcal{A}$ be Abelian categories with injective envelope, and $\mathcal{T}$ a Serre subcategory. Suppose the localisation sequence (lower part of 6-functors) of Serre subcategory.

$$\begin{array}{ccccccc}
\operatorname{Inj}(\mathcal{T}) & \longrightarrow & \operatorname{Inj}(\mathcal{A}) & \longrightarrow & \operatorname{Inj}(\mathcal{A}/\mathcal{T}) & \overset{\simeq}{\dashrightarrow} & \operatorname{Inj}(S^{\perp}) \\
\uparrow & & \uparrow & & \uparrow & & \vdots \\
\mathcal{T} & \underset{i^!}{\overset{i_*}{\rightleftarrows}} & \mathcal{A} & \underset{j_*}{\overset{j^{-1}}{\rightleftarrows}} & \mathcal{A}/\mathcal{T} & \overset{\simeq}{\dashrightarrow} & S^{\perp}
\end{array} \quad . \tag{4.33}$$

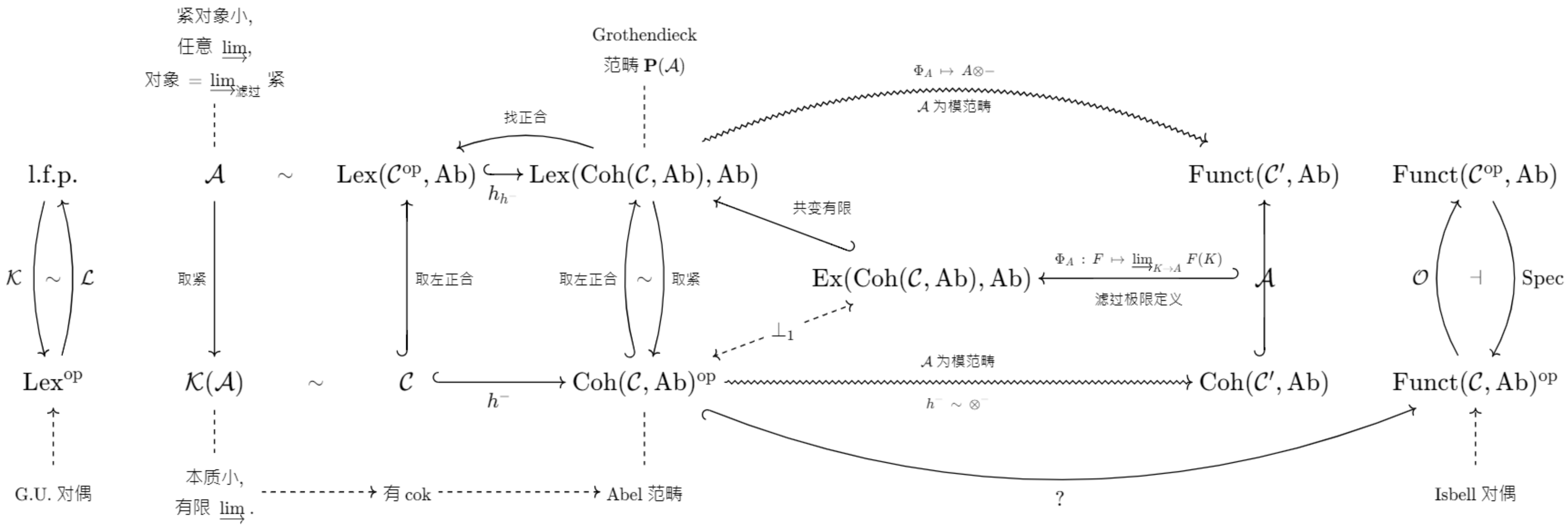
Then there is a functorial decomposition of injective objects, making

$$\operatorname{Spec}(\mathcal{T}) \sqcup \operatorname{Spec}(\mathcal{T}) \simeq \operatorname{Spec}(\mathcal{A}/\mathcal{T}). \tag{4.34}$$

Unwinding the topologies, the "spec" here means the isomorphism classes of injective envelopes. The ideal comes from Matlis duality.

Remark 4.29. $\mathcal{A}$ is usually a Grothendieck category, and $\mathcal{T}$ is a Serre subcategories closed under coproducts, thus is also a Grothendieck category.

Example 4.30. Where the Grothendieck categoryies usually found?


紧对象小,
任意 lim,
对象 = lim 滤过 紧
Grothendieck
范畴 P(A)
$\Phi_A \mapsto A\otimes -$
A 为模范畴
找正合
l.f.p.
A
~
Lex(C^op, Ab)
Lex(Coh(C, Ab), Ab)
h_h
Funct(C', Ab)
Funct(C^op, Ab)
共变有限
K
~
L
取紧
取左正合
取左正合
~
取紧
Ex(Coh(C, Ab), Ab)
$\Phi_A : F \mapsto \varinjlim_{K\to A} F(K)$
滤过极限定义
A
O
⊣
Spec
⊥1
Lex^op
K(A)
~
C
h
Coh(C, Ab)^op
A 为模范畴
Coh(C', Ab)
Funct(C, Ab)^op
$h^- \sim \otimes^-$
G.U. 对偶
本质小,
有限 lim.
有 cok
Abel 范畴
?
Isbell 对偶

4.3. Approximation, and more.

Example 4.31. Back to $\varphi : \mathcal{D} \to \mathcal{C}$. There is an induced

$$\varphi^{-1} : \mathrm{Ab}(\mathcal{C}) \to \mathrm{Ab}(\mathcal{D}), \quad G(-) \to G(\varphi(-)). \tag{4.35}$$

From axiom of choice, any module M admits some presentation $R^{\oplus\lambda} \to R^{\oplus\mu} \to M \to 0$. Hence we wish there are some "categorical colimit" for $F(-) = \varinjlim(-, X)$, such that there is $\varphi_! \dashv \varphi^{-1}$:

$$\begin{array}{ccc} \mathcal{C} & \xleftarrow{\quad\varphi\quad} & \mathcal{D} \\ \downarrow & \overset{\varphi_!}{\dashleftarrow} & \downarrow \\ \mathrm{Ab}(\mathcal{C}) & \xrightarrow[\varphi^{-1}]{} & \mathrm{Ab}(\mathcal{D}) \end{array} . \tag{4.36}$$

The isomorphism comes from

$$(\varinjlim(-, X), G(\varphi(-)))_{\mathrm{Ab}(\mathcal{C})} \simeq \varprojlim((-, X), G(\varphi(-)))_{\mathrm{Ab}(\mathcal{C})} \tag{4.37}$$

$$\overset{\text{Yoneda}}{\simeq} \varprojlim((-, \varphi(X)), G(-))_{\mathrm{Ab}(\mathcal{D})} \simeq (\varinjlim(-, \varphi(X)), G(-))_{\mathrm{Ab}(\mathcal{D})}. \tag{4.38}$$

Under "some condition", one has that

$$\begin{array}{ccccc} \mathcal{D} & \longrightarrow & \mathrm{ab}(\mathcal{D}) & \longrightarrow & \mathrm{Ab}(\mathcal{D}) \\ \varphi \downarrow & & \varphi_! \downarrow \;\; \uparrow \varphi^{-1} & & \varphi_! \downarrow \;\; \uparrow \varphi^{-1} \\ \mathcal{C} & \longrightarrow & \mathrm{ab}(\mathcal{C}) & \longrightarrow & \mathrm{Ab}(\mathcal{C}) \end{array} . \tag{4.39}$$

where the presentation of $\varphi_!(\mathrm{coker}(-, (f)))_{\mathcal{D}} = \mathrm{coker}(-, \varphi(f))_{\mathcal{C}}$. A condition is that φ is an inclusion of covariant finite subcategory, which state that $c \in \mathcal{C}$ admits a precover of some $d \to c$.

Under the settings above, we can complete the upper part of the picture:

$$\begin{tikzcd}
\mathcal{C}/\mathcal{D} \arrow[d, tail] & \mathcal{C} \arrow[l, tail, "i"'] \arrow[d, tail] & \mathcal{D} \arrow[l, tail, "j"'] \arrow[d, tail] \\
\mathrm{ad}(\mathcal{C}/\mathcal{D}) \arrow[r, tail, shift right, "i_*"'] & \mathrm{ab}(\mathcal{C}) \arrow[l, shift right, "i^{-1}"'] \arrow[r, shift right, "j^{-1}"'] & \mathrm{ab}(\mathcal{D}) \arrow[l, shift right, "j_!"']
\end{tikzcd} . \tag{4.40}$$

When all arrows exists, there are 6-commutative squares $\downarrow \circ k = K \circ \downarrow$.

Example 4.32. For Serre subcategory with good conditions, one has

$$\begin{tikzcd}
S^{\perp} \arrow[r, "i_*"] \arrow[d, dashed] & \mathcal{A} \arrow[l, dashed, bend right, "i^{-1}"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^{-1}"] \arrow[d, dashed] & \mathcal{T} \arrow[l, dashed, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \arrow[d, dashed] \\
\mathrm{ab}(S^{\perp}) \arrow[r, "I_*"] & \mathrm{ab}(\mathcal{A}) \arrow[l, bend right, "I^{-1}"'] \arrow[l, dashed, bend left, "I^!"] \arrow[r, "J^{-1}"] & \mathrm{ab}(\mathcal{T}) \arrow[l, bend right, "J_!"'] \arrow[l, dashed, bend left, "J_*"]
\end{tikzcd} . \tag{4.41}$$

Example 4.33. A ring is a degenerated category. Let R be a ring and with idenpotent element $e^2 = e$.

$$\begin{array}{c} \mathsf{Mod}_R \underset{({}_{eRe}(eR)_R,-)_R}{\overset{-\otimes_{eRe}(eR)_R}{\leftrightarrows}} \mathsf{Mod}_{eRe} \\ \mathsf{Mod}_{R^{\mathrm{op}}} \underset{(eR_{(eRe)^{\mathrm{op}}},-)}{\overset{eR\otimes_R -}{\rightleftarrows}} \mathsf{Mod}_{(eRe)^{\mathrm{op}}} \end{array} . \tag{4.42}$$

Now we generalise R to a additive categories $\mathcal{C}$ of Lex-type. Then fix any object $X \in \mathsf{Ob}(\mathcal{C})$, and take $\mathcal{D} : \overline{\{X\}}^{\oplus}$ (the projective objects generalised to X). One has the diagram

$$\begin{array}{ccccc} & \overset{j_!}{\dashrightarrow} & & \overset{i^{-1}}{\dashrightarrow} & \\ \mathcal{C}/\mathcal{D} & \xleftarrow{\ j^{-1}\ } & \mathcal{C} & \xleftarrow{\ i_*\ } & \mathcal{D} \\ & \underset{j_*}{\rightarrowtail} & & \underset{i^!}{\longrightarrow} & \\ \downarrow & & \downarrow & & \downarrow \\ & \overset{J_!}{\dashrightarrow} & & \overset{I^{-1}}{\dashrightarrow} & \\ \mathrm{ab}(\mathcal{C}/\mathcal{D}) & \xleftarrow{\ J^{-1}\ } & \mathrm{ab}(\mathcal{C}) & \xleftarrow{\ I_*\ } & \mathrm{ab}(\mathcal{D}) \\ & \underset{J_*}{\longrightarrow} & & \underset{I^{-1}}{\longrightarrow} & \end{array} . \tag{4.43}$$

There may be some problems in this diagram (but at least it is true for commutative Artinian rings, or for a big functor category Ab). We leave it as a conjecture.

4.4. Rerollement of Triangulated Categories (and its uses).

Definition 4.34 (Gluing (rerollement) of triangulated categires)**.** The data: three △ categories and six functors along with a diagram

$$\mathcal{V} \underset{i^!}{\overset{i^{-1}}{\leftarrow \!\!\xrightarrow{\ i_*\ }\!\! \leftarrow}} \mathcal{C} \underset{i_*}{\overset{j_!}{\leftarrow \!\!\xrightarrow{\ i^{-1}\ }\!\! \leftarrow}} \mathcal{W}\ , \tag{4.44}$$

satisfying the following conditions

(1) the left triple and right triple are adjoints;
(2) the composition of rows are zero;
(3) $j_!j^{-1}(\cdot) \to (\cdot) \to i_*i^{-1}(\cdot) \to j_!j^{-1}(\cdot)[1]$ is functorial from objects to distinguished triangles;
(4) $i_*i^{-1}(\cdot) \to (\cdot) \to j_*j^{-1}(\cdot) \to i_*i^{-1}(\cdot)[1]$ is functorial from objects to distinguished triangles;
(5) all red functors $j_!$, j_* and i_* are fully faithful, thus the remaining 4 units (counits) are isomorphisms.

For Serre subcategories, or sheaves in general, the exact sequences are only half-exact.

Proposition 4.35. *The exactness of rows:* $\ker(i^{-1}) = \operatorname{im}^1(j_!)$*, etc., follows from* △*.*

Remark 4.36*.* From series of works due to ?, the left (resp. right) part of the rerollement completes to the whole picture (unique under equivalence). There are also discussions on split rerollement (sheaves over component).

Example 4.37. (and its uses). Some remarkable works on rerollement of triangulated categories.

(1) (Happle's work on "finite dim gl conjecture") Let $D(B) \Longleftarrow D(A) \Longleftarrow D(C)$ be a recollement of derived categories over Artin algebras. $\operatorname{gl}\dim(A) < \infty$ (B has finite global dimension) whenever $\operatorname{gl}\dim(B) + \operatorname{gl}\dim(C) < \infty$.

(2) (A picture from Konig.) Assume there is rerollement of

$$D^-(B) \underset{i^!}{\overset{i^{-1}}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\quad i_* \quad}}}}} D^-(A) \underset{j_*}{\overset{j_!}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\quad j^{-1} \quad}}}}} D^-(C) \ . \tag{4.45}$$

(a) Assume either $\operatorname{gl}\dim(C) < \infty$ or $\operatorname{gl}\dim(C) < \infty$, the diagram restricts to D^b;

- Only lower half of the diagram exists in general.

(b) When $\operatorname{gl}\dim(C) < \infty$, the diagram restricts to $K^b(\operatorname{Proj}(-))$;

- Only upper half of the diagram exists in general.

(c) (♡) recollement of triangulated categories recollect t-structures.

$$\begin{array}{ccccc}
i^{-1}(X) & \xleftarrow{\ i^{-1}\ } & X & \xrightarrow{\ j^{-1}\ } & j^{-1}(X) \\
\downarrow & & \vdots & & \downarrow \\
\mathcal{A}^{\leq 0} & & \mathcal{C}^{\geq 0} & & \mathcal{B}^{\geq 0} \\
\mathcal{A} & \underset{i^!}{\overset{i^{-1}}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\quad i_* \quad}}}}} & \mathcal{C} & \underset{j_*}{\overset{j_!}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\quad j^{-1} \quad}}}}} & \mathcal{B} \\
\mathcal{A}^{\geq 0} & & \mathcal{C}^{\leq 0} & & \mathcal{B}^{\leq 0} \\
\uparrow & & \vdots & & \uparrow \\
i^!(Y) & \xleftarrow{\ i^!\ } & Y & \xrightarrow{\ j^{-1}\ } & j^{-1}(Y)
\end{array} \ . \tag{4.46}$$

Suppose that $\mathcal{A} \Longleftarrow \mathcal{B} \Longleftarrow \mathcal{C}$ are recollement, where the t-structures pf $\mathcal{A}$ and $\mathcal{C}$ are given. There is an induced t structure for $\mathcal{C}$, where

(i) $Y \in \mathcal{C}^{\leq 0}$ whenever $j^{-1}(Y) \in \mathcal{B}^{\leq 0}$ and $i^!(Y) \in \mathcal{A}^{\leq 0}$;

(ii) $X \in \mathcal{C}^{\geq 0}$ whenever $j^{-1}(X) \in \mathcal{B}^{\geq 0}$ and $i^{-1}(X) \in \mathcal{A}^{\geq 0}$;

Definition 4.38 (t-structure)**.** Let $(D, [1], \triangle)$ be a triangulated category.

One can simply view D as a derived category of a Abelian category.

A t-structure of on D is a pair of full subcategory $(D^{\leq 0}, D^{\geq 0})$,

A simple example: $D^I = \mathrm{Ch}^I(\mathcal{A})/\sim$, where $X \in \mathrm{Ch}^I(\mathcal{A}) \iff \{H^i(X) = 0 \mid i \notin S\}$.

such that

(1) $D^{\geq 0}$ is closed under suspension $[1]$, and $D^{\leq}$ is closed under desuspension $[-1]$.

(2) $(-,-) : D^{\leq 0} \times D^{\geq 1} \to 0$ (or simply $(-\infty, 0] \cap [1, +\infty) = \emptyset$).

(3) Every object X admits a t-factorisation $(-)_{\leq 0} \to X \to (-)_{\geq 1} \to (-)_{\leq 0}[1]$.

Proposition 4.39. *The heart $D^{\heartsuit} := D^{\geq 0} \cap D^{\leq 0}$ is an Abelian category.*

For simple cases of derived categories, $\mathcal{A} \hookrightarrow \mathrm{D}^0\mathcal{A}$ is fully faithful.

By support of homological groups, it is easy to guess

$$\ker(f) = (\ker^{\triangle}(f))_{\leq 0}, \quad \operatorname{coker}(f) = (\operatorname{coker}^{\triangle}(f))_{\geq 0}. \tag{4.47}$$

The amazing part is that $\mathrm{coim} \simeq \mathrm{im}$ *is due to octagon lemma.*

Remark 4.40 (stable t-structure)*.* Say a t-structure $(D^{\leq}, D^{\geq})$ is stable, if both $D^{\geq,\leq}$ are triangulated. In short, stable means closed under $[\pm 1]$. The “combinatorial” examples (e.g. the derived category) are not stable, thus we never use the notation $D^{\geq,\leq}$ for stable t-structures thenceforth.

A series of examples of stable triangulated categories come from projective-acyclic pairs.